On Quadratic Convergence of Successive Iterative Method in Nonlinear Two-point Boundary Value Problem

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1.

Let us examine the linear equation

$$u''=0 \quad , \quad u(0)=a_1 \qquad u(b)=a_2 \tag{1.1}$$

a two-point boundary value problem.

Let u_1 and u_2 be the two principal solutions of the homogeneous equation above, defined by the initial conditions

$$u_1(0)=1 \qquad u_2(0)=0 \qquad u'_1(0)=0 \qquad u'_2(0)=1. \tag{1.2}$$

Then, by virtue of the linearity of (1.1), we have

$$u(t)=a_1u_1(t)+\frac{a_2-a_1u_1(b)}{u_2(b)}u_2(t).$$

By the initial conditions (1.2), we have

$$u_1=1 \qquad u_2=t. \tag{1.3}$$

Hence, we can see that

$$u=a_1+\frac{a_2-a_1}{b}\,t.$$

We now turn our attention to the inhomogeneous equation

$$u''+r(t)=0$$

with the two-point boundary conditions

$$u(0)=a_1 \qquad u(b)=a_2.$$

Taking advantage of linearity, we write

$$u=v+w$$

where w and v are chosen, respectively, to satisfy the equations

$$\begin{aligned} &w''+r(t)=0, \qquad w(0)=0 \qquad w'(0)=0 \\ \text{and} \quad &v''=0, \qquad v(0)=a_1 \qquad v(b)=a_2-w(b). \end{aligned} \tag{1.4}$$

Let u_1 and u_2 be particular solutions of the homogeneous equation (1.1), and write

$$w=s_1u_1+s_2u_2$$

where s_1 and s_2 are functions of t to be determined at our convenience.

Then

$$w'=s_1u'_1+s_2u'_2+s'_1u_1+s'_2u_2.$$

To simplify, set

$$s'_1u_1+s'_2u_2=0. \tag{1.5}$$

Since $w'=s_1u'_1+s_2u'_2$, with this condition on s'_1 and s'_2, we have

$$w''=s_1u''_1+s_2u''_2+s'_1u'_1+s'_2u'_2.$$

Since u_1 and u_2 are the principal solutions of the homogeneous equation (1.1),

$$u''_1 = 0 \quad \text{and} \quad u''_2 = 0.$$

Hence, combining with (1.4),

$$s'_1 u'_1 + s'_2 u'_2 = -r(t).$$

Combining this with (1.5), we have two simultaneous linear algebraic equations for s'_1 and s'_2.

A solution exists, provided that the determinant

$$W(t) = \begin{vmatrix} u_1 & u_2 \\ u'_1 & u'_2 \end{vmatrix}$$

is nonzero. From (1.3) we have

$$W(t) = \begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix} = 1$$

and

$$s'_1 = \frac{\begin{vmatrix} 0 & u_2 \\ -r(t) & u'_2 \end{vmatrix}}{W(t)} = \begin{vmatrix} 0 & t \\ -r(t) & 1 \end{vmatrix} = tr(t)$$

$$s'_2 = \frac{\begin{vmatrix} u_1 & 0 \\ u'_1 & -r(t) \end{vmatrix}}{W(t)} = \begin{vmatrix} 1 & 0 \\ 0 & -r(t) \end{vmatrix} = -r(t).$$

Hence,

$$s_1 = \int_0^t t_1 r(t_1) dt_1$$

$$s_2 = -\int_0^t r(t_1) dt_1$$

choosing $s_1(0) = s_2(0) = 0$, since we want to satisfy the conditions $w(0) = w'(0) = 0$.

Thus,

$$w = \int_0^t r(t_1)(t_1 - t)\, dt_1 = \int_0^t G(t, t_1) r(t_1) dt_1$$

where $G(t, t_1) = t_1 - t$. (1.6)

Now we shall solve the equation

$$u'' + r(t) = 0, \qquad u(0) = u(b) = 0. \tag{1.7}$$

The general solution of (1.7) has the form

$$u = c_2 u_2(t) + \int_0^t G(t, t_1) r(t_1) dt_1$$

where c_2 is a constant to be determined. Setting $t = b$, we see that

$$c_2 = -\int_0^b \frac{G(b, t_1) r(t_1)}{u_2(b)} dt_1.$$

Hence,

$$u = \frac{-u_2(t)}{u_2(b)} \int_0^b G(b, t_1) r(t_1) dt_1 + \int_0^t G(t, t_1) r(t_1) dt_1$$

$$= \frac{-u_2(t)}{u_2(b)} \int_t^b G(b, t_1) r(t_1) dt_1 + \int_0^t \left[G(t, t_1) - \frac{u_2(t)}{u_2(b)} G(b, t_1) \right] r(t_1) dt_1.$$

Thus,

$$u(t)=\int_0^b K(t,\ t_1,\ b)r(t_1)dt_1$$

where

$$K(t,\ t_1,\ b)=\begin{cases} G(t,\ t_1)-\dfrac{u_2(t)}{u_2(b)}G(b,\ t_1) & o\leqq t_1\leqq t \\ -\dfrac{u_2(t)}{u_2(b)}G(b,\ t_1) & t\leqq t_1\leqq b. \end{cases}$$

By (1.6)

$$K(t,\ t_1,\ b)=\begin{cases} t(b-t_1)/b & t\leqq t_1\leqq b \\ t_1(b-t)/b & o\leqq t_1\leqq t. \end{cases}$$

2.

Now we apply quasilinearization to the two-point boundary value problem,

$$u''+f(u',\ u,x)=0$$
$$u(0)=u(b)=0.$$

Let $u_0(x)$ be some initial approximation and consider the sequence $\{u_n\}$ determined by the recurrence relation

$$u_{n+1}''+f_{u'}(u_n',\ u_n,\ x)(u_{n+1}'-u_n')+f_u(u_n',\ u_n,\ x)(u_{n+1}-u_n)+f(u_n',\ u_n,\ x)=0$$
$$u_{n+1}(0)=u_{n+1}(b)=0. \tag{2.1}$$

We obtain the linear integral equation

$$u_{n+1}=\int_0^b K(x,\ y)[f(u_n',\ u_n,\ y)+f_u(u_n',\ u_n,\ y)(u_{n+1}-u_n)+f_{u'}(u_n',\ u_n,\ y)(u_{n+1}'-u_n')]dy \tag{2.2}$$

where K(x, y) is the Green's function

$$K(x,\ y)=\begin{cases} (b-x)y/b & o\leqq y\leqq x \\ (b-y)x/b & x\leqq y\leqq b. \end{cases}$$

We can easily see that

$$\max_{x,\ y} K(x,\ y)=\frac{b}{4}$$

where the maximization is over the region $0\leqq x,\ y\leqq b$.

Let

$$\max_{|u|\leqq 1} \max\ (|f(u',\ u,\ x)|,\ |f_u(u',\ u,\ x)|,\ |f_{u'}(u',\ u,\ x)|)=m,$$

assuming that $m<\infty$, and choose $u_0(x)$ so that

$$|u_0(x)|\leqq 1,\ \text{for}\ 0\leqq x\leqq b.$$

Turning to (2.2), we have

$$|u_{n+1}|\leqq\int_0^b |K(x,\ y)|[|f(u_n',\ u_n,\ y)|+|f_u(u_n',\ u_n,\ y)||u_{n+1}|+|f_u(u_n',\ u_n,\ y)||u_n|$$
$$+|f_{u'}(u_n',\ u_n,\ y)||u_{n+1}'|+|f_{u'}(u_n',\ u_n,\ y)||u_n'|]dy.$$

Hence, writing $m_1=\max_{o\leqq x\leqq b}|u_1(x)|$ and assuming $|u_n'(x)|\leqq c|u_n(x)|$ $(n=0,\ 1,\ 2,\cdots)$

for some finite number c, we have, for $n=0$

$$m_1 \leqq \frac{b}{4}\int_0^b [m + mm_1 + m + mcm_1 + mc]\, dy = \frac{b^2m(2+c)}{4} + \frac{b^2(1+c)mm_1}{4}.$$

Provided, therefore, that $\frac{b^2(1+c)m}{4} < 1$, we obtain the bound

$$m_1 \leqq \frac{\frac{b^2m(2+c)}{4}}{1-\frac{b^2(1+c)m}{4}}.$$

The upper bound is itself less than 1 if $b^2 \leqq \frac{4}{m(3+2c)}$, and we can establish the uniform boundedness of the sequence $\{u_n(x)\}$ for b sufficiently small.

We have thus demonstrated that the inductive definition of the sequence $\{u_n(x)\}$ is meaningful.

We shall show that this sequence $\{u_n\}$ converges quadratically.

Returning to the recurrence relation of (2.1), let us subtract the n-th equation from the (n+1) st

$$\begin{aligned}(u_{n+1}-u_n)'' + f(u_n', u_n, x) - f(u_{n-1}', u_{n-1}, x)\\ -(u_n-u_{n-1})f_u(u_{n-1}', u_{n-1}, x) - (u_n'-u_{n-1}')f_{u'}(u_{n-1}', u_{n-1}, x)\\ +(u_{n+1}-u_n)f_u(u_n', u_n, x) + (u_{n+1}'-u_n')f_{u'}(u_n', u_n, x) = 0.\end{aligned}$$

Regarding this as a differential equation for $u_{n+1}-u_n$ and converting into an integral equation as before, we have

$$\begin{aligned}u_{n+1}-u_n = \int_0^b K(x, y)[f(u_n', u_n, y) - f(u_{n-1}', u_{n-1}, y)\\ -(u_n-u_{n-1})f_u(u_{n-1}', u_{n-1}, y) - (u_n'-u_{n-1}')f_{u'}(u_{n-1}', u_{n-1}, y)\\ +(u_{n+1}-u_n)f_u(u_n', u_n, y) + (u_{n+1}'-u_n')f_{u'}(u_n', u_n, y)]\, dy.\end{aligned}$$

The mean-value theorem tells us that

$$\begin{aligned}&f(u_n', u_n, x) - f(u_{n-1}', u_{n-1}, x)\\ &\quad -(u_n-u_{n-1})f_u(u_{n-1}', u_{n-1}, x) - (u_n'-u_{n-1}')f_{u'}(u_{n-1}', u_{n-1}, x)\\ &= \frac{1}{2}\{(u_n-u_{n-1})^2\frac{\partial^2}{\partial u^2}f(\Theta_1, \Theta_2, x) + 2(u_n-u_{n-1})(u_n'-u_{n-1}')\frac{\partial^2}{\partial u\partial u'}f(\Theta_1, \Theta_2, x)\\ &\quad + (u_n'-u_{n-1}')^2\frac{\partial^2}{\partial u'^2}f(\Theta_1, \Theta_2, x)\}\end{aligned}$$

where Θ_1 lies between u_{n-1} and u_n, and Θ_2 lies between u_{n-1}' and u_n'.

Hence, letting

$$k = \max_{|u|\leqq 1} \max[(|\frac{\partial^2}{\partial u^2}f(u', u, x)|, |\frac{\partial^2}{\partial u\partial u'}f(u', u, x)|, |\frac{\partial^2}{\partial u'^2}f(u', u, x)|),$$

we have, very much as before,

$$\begin{aligned}|u_{n+1}-u_n| \leqq \frac{b}{4}\int_0^b [\frac{k}{2}\{(u_n-u_{n-1})^2 + 2c(u_n-u_{n-1})^2\\ + c^2(u_n-u_{n-1})^2\} + m|u_{n+1}-u_n| + mc|u_{n+1}-u_n|]\, dy.\end{aligned}$$

Hence,

$$\max_x |u_{n+1}-u_n| \leqq \frac{\frac{b^2k(1+c)^2}{8}}{1-\frac{b^2m(1+c)}{4}}(\max_x|u_n-u_{n-1}|)^2.$$

This shows that there is quadratic convergence if there is convergence at all. This convergence depends upon the quantity

$$\frac{\frac{b^2k(1+c)^2}{8}}{1-\frac{b^2m(1+c)}{4}}(\max_x|u_1-u_0|)$$

which by the standard procedures can be shown to be less than one for b sufficiently small.

And it is also seen that even if the interval [0, b] appears to be too large initially, it is sufficient for convergence that $\max_x|u_1(x)-u_0(x)|$ is sufficiently small.

References

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