Σ-SYMMETRIC SPACES AND THE CLOSED GRAPH THEOREM

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Concerning to the study of the closed graph theorem, T. Husain [1] discussed the $B(\mathfrak{C})$-spaces defined as follows:

Definition 1. *Let $\mathfrak{C}$ denote a fixed class of locally convex Hausdorff topological (abbreviated to l. c.) spaces. An l. c. space E is said to be a $B(\mathfrak{C})$-space if, for each l. c. space $F \in \mathfrak{C}$, a linear continuous and almost open mapping f of E onto F is open. If f is one-to-one, then E is called a $Br(\mathfrak{C})$-space.*

In particular, when $\mathfrak{C}$ is the class of all l. c. spaces, a $B(\mathfrak{C})$-space is nothing more than a B-complete (fully complete) space [4]. In [1] T. Husain discussed the properties of $B(\mathfrak{T})$-spaces, where $\mathfrak{T}$ denotes the class of all barrelled spaces.

The purpose of this note is to show how the theorems of [1] and [2] can be more extensively applied to the case where $\mathfrak{C}$ is a class of Σ-symmetric spaces.

First we need some definitions and simple results.

1. Let E_u be a linear space E endowed with the topology u and let Σ be a class of bounded subsets of E_u whose union is E. Let E' denote the topological dual of E_u. and let E'_Σ be the linear space E' with the l. c. topology of uniform convergence on all members of Σ (abbreviated to Σ-topology). E', considered as an l. c. space, denotes the dual of E with $\sigma(E', E)$-topology.

Definition 2. *An l. c. space E_u is said to be Σ-symmetric if any of the following equivalent conditions hold:*

(1) *Every barrel in E_u which absorbs every member of Σ is a neighbourhood of zero.*

(2) *Every bounded subset of E'_Σ is equicontinuous.*

(3) *The topology induced on E by the strong dual of E'_Σ is the original topology of E_u.*

(4) *E_u has the relatively strong topology $\tau(E', E)$, and every convex bounded subset of E'_Σ has compact closure in E'.*

The equivalence of these conditions was proved in [5]. If $\Sigma_1 \subset \Sigma_2$, it is clear that Σ_1-symmetry implies Σ_2-symmetry; the strongest restriction on E_u is obtained by taking for Σ the class of all subsets of E_u consisting of a single point, and then Σ-symmetry is simply the property of being barrelled. If Σ is the class of all bounded subsets of E_u, the Σ-topology is called the strong topology on E' and we have the weakest Σ-symmetric property, which is that of being quasi-barrelled.

Let E_u and F_v be two l. c. spaces. Let f be a linear continuous mapping of E_u onto F_v. Then, if Σ is a class of bounded subsets of E_u whose union is E, the class $f(\Sigma)$ of all $f(S)$, $S \in \Sigma$ is also that of bounded subsets of F_v whose union is F. Therefore we can define on F' the $f(\Sigma)$-topology.

Lemma 1. *Let E_u and F_v be two l.c. spaces. Let f be a linear continuous and almost open mapping of E_u onto F_v. If E_u is Σ-symmetric, F_v is $f(\Sigma)$-symmetric.*

Proof. Let B be a barrel in F_v which absorbs every member of $f(\Sigma)$. Since f is linear, $f^{-1}(B)$ is convex and circled, and it absorbs every member of Σ. Further, as f is continuous $f^{-1}(B)$ is also closed. Therefore $f^{-1}(B)$ is a u-neighbourhood of 0, because E_u is a Σ-symmetric space by hypothesis. Now as f is almost open, $\overline{f(f^{-1}(B))}=\overline{B}=B$ is a v-neighbourhood of 0. This proves the lemma.

Corollary 1. *Let u and v be two l.c. tepologies on a same linear space E and let i be the identity mapping of E_u onto E_v. Further let i be continuous and almost open. Then, if E_u is Σ-symmetric, so is E_v.*

Corollary 2. *Let M be a closed subspace of an l.c. space E_u and let φ be the natural mapping of E_u onto E_u/M. Then if E_u is Σ-symmetric, E_u/M is $\varphi(\Sigma)$-symmetric.*

Now we define the $B(\Sigma)$-spaces as follows:

Definition 3. *Let E_u be an l.c. space and let Σ be a class of bounded subsets of E_u whose union is E. Then E_u is said to be a $B(\Sigma)$-space if and only if every continuous and almost open mapping f of E_u onto an $f(\Sigma)$-symmetric space F_v is open. If f is one-to-one then E_u is called a $Br(\Sigma)$-space.*

In particular, when Σ is the class of all subsets of E consisting of a single point, then obviously a $B(\Sigma)$-space is nothing more than a $B(\mathfrak{T})$-space. Therefore our notion of $B(\Sigma)$-spaces contains as a particular case that of $B(\mathfrak{T})$-spaces.

2. In [4], a characterization of a B-complete space in terms of the subspace of its dual was given. We give in this section a similar characterization of a $B(\Sigma)$-space.

A set H in the dual E' of an l.c. space E_u is equicontinuous if and only if there is a u-neighbourhood U of 0 in E_u such that $H \subset U^\circ$. Let Q be a subspace of E' with the relative $\sigma(E',E)$-topology. A set H in Q is said to be equicontinuous if it is equicontinuous in E'.

Definition 4. *A subspace Q of E', when Q is endowed with the relative $\sigma(E',E)$-topology, is said to be Σ-boundedly complete if the following conditions are satisfied:*

(a) *Q is almost closed, i.e. for each u-neighbourhood U of 0 in E_u. $Q \cap U^\circ$ is closed in E'.*

(b) *Every bounded set of Q with respect to the relative Σ-tupology (abbreviated to a Σ-bounded set) is equicontinuous.*

Then it is easy to show that every Σ-boundedly complete subspace is quasi-complete. Now we get the following theorems.

Theorem 1. *A necessary and sufficient condition for an l.c. space E_u to be a $Br(\Sigma)$-space is that each dense Σ-boundedly complete subspace Q of E' coincides with E'.*

Proof. For the 'necessary' part, assume that E_u is a $Br(\Sigma)$-space. The density of Q implies $Q'=E$. Let v denote the $\mathfrak{S}$-topology on E, where $\mathfrak{S}$ consists of all Σ-bounded sets of Q. First we show that $v=\tau(E,Q)$. Since the topology $\sigma(Q,E)$ coincides with $\sigma(E',E)$ on Q, a $\sigma(Q,E)$-compact convex circled subset C of Q is $\sigma(E',E)$-compact in E'. Therefore C is strongly bounded in E' (see, for example, [3] p. 170, 18.5) and hence Σ-bounded. In other words, the class of all $\sigma(Q,E)$-

compact convex circled subsets of Q is included in $\mathfrak{S}$. Hence $v \supset \tau(E, Q)$. On the other hand, according to (a) of Definition 3, for each convex circled u-neighbourhood U of 0, $(Q \cap U^{\circ})^{\circ}$ is a $\tau(E, Q)$-neighbourhood of 0. But according to (b) of Definition 3, for each Σ-bounded set B of Q, $B \subset Q \cap U^{\circ}$ implies $B^{\circ} \supset (Q \cap U^{\circ})^{\circ}$. This shows that $v \subset \tau(E, Q)$. Combining the two inclusion relations we have $v = \tau(E, Q)$. Since Q is quasi-complete, it follows by (4) of Definition 2 that E_v is a Σ-symmetric space. Now we show that $u \supset v$. For each v-neighbourhood V of 0, there exists a Σ-bounded set B of Q such that $B^{\circ} \subset V$. But as Q is Σ-boundedly complete, (b) of Definition 4 implies $B \subset U^{\circ}$ for some u-neighbourhood U of 0 in E. But then $V \supset B^{\circ} \supset U^{\circ\circ} \supset U$ implies $u \supset v$. This shows the continuity of the identity mapping i of E_u onto E_v. Furthermore, for each convex circled u-neighbourhood U of 0, $\overline{i(U)}$ is a barrel in E_v which absorbs every member of Σ, and is therefore a v-neighbourhood of 0 by the Σ-symmetry of E_v. This shows the almost openness of i. Consequently i is a continuous and almost open mapping of E_u onto a Σ-symmetric space E_v. Hence i is open because E_u is a $Br(\Sigma)$-space. Therefore $u = v$ and hence $Q = E_v' = E_u'$. The 'sufficient' part is an obvious modification of the proof of Theorem 1 of [1].

A similar characterization for $B(\Sigma)$-spaces is as follows.

Theorem 2. *A necessary and sufficient condition for an l.c. space E_u to be a $B(\Sigma)$-space is that each Σ-boundedly complete subspace Q of the dual E' of E is $\sigma(E', E)$-closed.*

Proof. For the 'necessary' part, let φ be the natural mapping of E onto $Q' = E/Q^{\circ}$. Let v denote the $\mathfrak{S}$-topology on Q', where $\mathfrak{S}$ consists of all Σ-bounded sets of Q. As shown in Theorem 1, Q'^{v} is $\varphi(\Sigma)$-symmetric because Q is quasi-complete and $v = \tau(Q', Q)$. The continuity of the mapping φ: $E_u \to Q'^{v}$ can be shown in the same way as in the proof of Theorem 2 of [1]. Furthermore, for each convex circled u-neighbourhood U of 0, $\overline{\varphi(U)}$ is a barrel in Q'^{v} which absorbs every member of $\varphi(\Sigma)$ and is therefore a v-neighbourhood of 0 by the $\varphi(\Sigma)$-symmetry of Q'^{v}. This shows the almost openness of φ. Then, as E_u is a $B(\Sigma)$-space, φ is open. In other words, $Q = Q^{00}$. This implies Q is $\sigma(E'. E)$-closed. The 'sufficient' part is again an obvious modification of the proof of Theorem 2 of [1].

3. In this last section we shall show the closed graph theorem for $Br(\Sigma)$-spaces, First we state the following theorem for a time.

Theorem 3. *Let F_v be an l.c. space and E_u a Σ-symmetric $Br(\Sigma)$-space. Let f be a linear mapping of F_v into E_u, the graph of which is closed in $F \times E$. If f is almost continuous, then f is continuous.*

Proof. Making use of the Corollary to Lemma 1, we get the proof as an obvious modification of the proof of Theorem 5 of [1].

But this theorem is merely a different version of Theorem 3.8 of [4] as is shown by the next Lemma 2. The circumstance is the same as was pointed out in [2].

Lemma 2. *An l.c. space E_u which is both a Σ-symmetric space and a $Br(\Sigma)$-space is Br-complete.*

Proof. Let Q be an almost closed dense subspace of the dual E' of E_u. Let B be a Σ-bounded subset of Q, where Q is endowed with the relative $\sigma(E',E)$-topology. Then B is $\sigma(E',E)$-bounded in E' and hence equicontinuous since E is Σ-symmetric. This shows that Q is Σ-boundedly complete. But then E being a $Br(\Sigma)$-space, it follows that Q is $\sigma(E',E)$-closed, and therefore E_u is Br-complete due to the characterization of the latter space in [4].

The following theorem is an improved one.

Theorem 4. *Let E_u be a Σ-symmetric space and F_v a $Br(f(\Sigma))$-space. Let f be a linear mapping of E_u onto F_v the graph of which is closed in $E \times F$. If f is almost open and almost continuous then f is continuous.*

Proof. We give an outline of the proof. For the detail, see the corresponding proof of Theorem 3 of [2]. Let $\{V\}$ denotes a fundamental system of closed convex and circled neighbourhoods of 0 in F_v. For each V in $\{V\}$, let $\overset{*}{V} = f(\overline{f^{-1}(V)})$. Then $\{\overset{*}{V}\}$ forms a fundamental system of neighbourhoods of 0 in F under an l. c. topology w (The closedness of the graph of f implies that w is Hausdorff). Since the mapping $f : E_u \to F_w$ is known to be continuous and almost open, F_w is a $f(\Sigma)$-symmetric space by Lemma 1. On the other hand, the identity mapping $i : F_v \to F_w$ is also continuous and almost open. Therefore, it follows that i is open, because F_v is a $Br(f(\Sigma))$-space. Hence $v = w$. Since $f : E_u \to F_w$ has been proved to be continuons, $f : E_u \to F_v$ is continuous.

We get from this theorem the following one.

Theorem 5. *Let E_u be a Σ-symmetric space and let g be a linear almost continuous and almost open mapping of an l. c. space F_v onto E_u with the closed graph. If $F_v/g^{-1}(O)$ is a $Br(\Sigma)$-space, then g is open.*

Proof. It can be assumed that g is one-to-one. Hence $g^{-1} : E_u \to F_v$ exists and it is almost continuous and almost open, because g is almost open and almost continuous. Therefore Theorem 4 is applied and hence g^{-1} is continuous. This shows that g is open.

The following theorems of [2] are immediate corollaries of Theorems 4 and 5.

Theorem 6. *Let E_u be a barrelled space and F_v a $Br(\mathfrak{T})$-space. Let f be a linear mapping of E_u onto F_v with the closed graph. If g is almost open, then f is continuous.*

For the proof, notice that a linear mapping of a barrelled space onto an l. c. space is always almost continuous.

Theorem 7. *Let E_u be a barrelled space and F_v a $Br(\mathfrak{T})$-space. Then a linear almost continuous mapping g of F_v onto E_u with the closed graph is open.*

For the proof, notice that a linear mapping of an l. c. space E onto a barrelled space F is always almost open and that a quotient space of a $Br(\mathfrak{T})$-space is also a $Br(\mathfrak{T})$-space.

The last theorem is another characterization of $Br(\Sigma)$-spaces.

Theorem 8. *Let E_u be a Σ-symmetric space. Then E_u is a $Br(\Sigma)$-space if and only if any one-to-one linear and almost open mapping f of E onto a $f(\Sigma)$-symmetric space F_v with the closed graph is open.*

Proof. Assume E_u is a $Br(\Sigma)$-space. Then according to Lemma 2 and Theorem (3.6) of [4], f is open. On the other hand, if f is a one-to-one continuous and almost open mapping of E_u onto any $f(\Sigma)$-symmetric space F_v, then the graph of f is closed in $E \times F$ and therefore f is open by assumption. Therefore E_u is a $Br(\Sigma)$-space.

References

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