On the (p_0, p)-Asymptotic Stability of the System of Differential Equations

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1. Introduction

It is well-known that, in 1892, A. M. Liapunov discussed the stability of solutions of the system of ordinary differential equations by utilizing the scalar function satisfying certain conditions.

Liapunov's second method is a very useful and powerful instrument in discussing the stability. However, even though the construction of the Liapunov function $V(t, x)$ is an art, it is difficult to find the Liapunov function satisfying some conditions. Therefore, it is important to obtain the weak sufficient condition for the stability theorem.

The stability property can be considered as a family of properties depending on some parameters. Consequently, when we employ a single Liapunov function to prove a given stability property, the Liapunov function used is assumed to play the role for every choice of these parameters. As a result, if we utilize a family of Liapunov functions instead of one, it is natural to expect that each member of the family has to satisfy weaker conditions. This is a precise idea using a family of Liapunov functions and the advantage is more clearly seen in the case of uniform stability properties. (cf. [3], [7], [10])

There are several different concepts of stability. To unify these varieties of stability notions and offer a general framework for investigation, it is convenient to introduce stability concepts in terms of two different measures.

In [6], the idea of perturbing Liapunov functions is introduced which is useful in the study of non uniform stability under weaker conditions.

In 1989, V. Lakschmikantham and Xin Zhi Liu have discussed the new non uniform stability, which they called the (p_0, p)-stability and the (p_0, p)-asymptotic stability, employing perturbing families of Liapunov functions.

In this paper, by using Liapunov's second method, we will state some generalization of the sufficient conditions for the (p_0, p)-asymptotic stability of the system of ordinary differential equations.

2. Notations and Definitions

First, we summarize some basic notations and definitions we will need later on.

Let I denote the interval $0 \leqq t < \infty$, R^n denote Euclidean n-space. For $x \in R^n$, let $\|x\|$ be any norm of x and denote by S_H the set of x such that $\|x\| < H$, $H > 0$.

We shall denote by $C(I \times R^n, R^n)$ the set of all continuous functions defined on $I \times R^n$ valued in R^n.

Let us list the following classes of functions for convenience.

$K = \{\sigma \in C(I, I)\ ;\ \sigma(r)$ is strictly increasing and $\sigma(0) = 0\}$.

$CK = \{\sigma \in C(I \times I, I)\ ;\ \sigma(t, r) \in K$ for each $t \in I\}$.

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$\Gamma = \{p \in C(I \times R^n, R)\ ;\ \inf_{x \in R^n} p(t, x) = 0$ for each $t \in I\}$.
$S(p, \rho) = \{(t, x)\ ;\ p(t, x) < \rho\}$.

We consider the system of differential equations

(1) $x' = f(t, x)$,

where $f \in C(S(p, \rho), R^n)$ and $f(t, 0) \equiv 0$.

Suppose that $f(t, x)$ is smooth enough to ensure existence, uniqueness and continuous dependence of solutions of the initial value problem.

Throughout this paper, the solution through a point $(t_0, x_0) \in I \times R^n$ will be denoted by such a form as $x(t, t_0, x_0)$.

[**Definition 1**] *Let $p_0, p \in \Gamma$. Then we say that p_0 is finer than p if there exist a $\rho > 0$ and a function $\phi \in K$ such that $p_0(t, x) < \rho$ implies $p(t, x) \leqq \phi(p_0(t, x))$.*

[**Definition 2**] *The system* (1) *is said to be (p_0, p)-stable if any $\varepsilon > 0$ and any $t_0 \in I$, there exists a $\delta(t_0, \varepsilon) > 0$ such that $p_0(t_0, x_0) < \delta(t_0, \varepsilon)$ implies $p(t, x(t, t_0, x_0)) < \varepsilon$ for all $t \geqq t_0$.*

[**Definition 3**] *The system* (1) *is said to be (p_0, p)-asymptotically stable, if the system* (1) *is (p_0, p)-stable and if there exists a $\delta_0(t_0) > 0$ such that if $p_0(t_0, x_0) < \delta_0(t_0)$, $p(t, x(t, t_0, x_0)) \to 0$ as $t \to \infty$.*

[**Definition 4**] *The zero solution of* (1) *is said to be stable if for any $\varepsilon > 0$ and any $t_0 \in I$, there exists a $\delta(t_0, \varepsilon) > 0$ such that $\|x_0\| < \delta(t_0, \varepsilon)$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geqq t_0$.*

[**Definition 5**] *The zero solution of* (1) *is said to be uniformly stable if δ of Definition 4 is independent of t_0.*

For $V \in C(I \times R^n, I)$, we define the function

$$\dot{V}(t, x)_{(1)} = \limsup_{h \to +0} \frac{1}{h}\{V(t+h, x+hf(t, x)) - V(t, x)\}.$$

In case, $V(t, x)$ has continuous partial derivative of the first order, it is evident that

$$\dot{V}(t, x)_{(1)} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t, x),$$

where "•" denotes an inner product.

[**Definition 6**] *Let $V \in C(I \times R^n, I)$ and $p \in \Gamma$. Then V is said to be weakly p-decrescent if there exist a $\rho_0 > 0$ and a function $\phi \in CK$ such that*

$p(t, x) < \rho_0$ implies $V(t, x) \leqq \phi(t, p(t, x))$.

Remark : If $p(t, x) = \|x\|$, $p_0(t, x) = \|x\|$, *then the* (p_0, p)-stability reduces to the well-known stability of the zero solution, and the (p_0, p)-asymptotic stability reduces to asymptotic stability of the zero solution.

Also, if $p(t, x) = \|x\|_k$ $(1 \leqq k < n$; k, n : positive integers), $p_0(t, x) = \|x\|$, then the (p_0, p)-stability is the partial stability of the zero solution, where $\|x\|_k$ is a norm of some partial components of x.

3. Preliminary Results

In 1989, V. Lakschmikantham and Xin Zhi Liu gave a sufficient condition for the (p_0, p)-stability and the (p_0, p)-asymptotic stability. We repeat these theorems.

【Theorem 3. 1】 *Assume that*

(A1) $p_0, p \in \Gamma$ *and* p_0 *is finer than* p,

(A2) $V_1 \in C(S(p, \rho), I)$, $V_1(t, x)$ *is locally Lipschitzian in x, weakly* p_0*-decrescent, and*

$$\dot{V}_1(t, x)_{(1)} \leqq g_1(t, V_1(t, x)),$$

where $g_1 \in C(I \times I, R)$ *with* $g_1(t, 0) \equiv 0$,

(A3) *for any* $\eta > 0$, *there exists a* $V_{2\eta} \in C(S(p, \rho) \cap S^c(p_0, \eta), I)$, *where* $S^c(p_0, \eta)$ *is the complement of* $S(p_0, \eta)$, $V_{2\eta}(t, x)$ *is locally Lipschitzian in x,*

$$a(p(t, x)) \leqq V_{2\eta}(t, x) \leqq b(p_0(t, x)) \text{ on } S(p, \rho) \cap S^c(p_0, \eta),$$

where $a, b \in K$ *and*

$$\dot{V}_1(t, x)_{(1)} + V_{2\eta}(t, x)_{(1)} \leqq g_2(t, V_1(t, x) + V_{2\eta}(t, x)) \text{ on } S(p, \rho) \cap S^c(p_0, \eta),$$

where $g_2 \in C(I \times I, R)$ *with* $g_2(t, 0) \equiv 0$,

(A4) *the zero solution of the scalar differential equation*

$$u' = g_1(t, u), \ u(t_0, t_0, u_0) = u_0 \geqq 0$$

is stable, and the zero solution of the scalar differential equation

$$v' = g_2(t, v), \ v(t_0, t_0, v_0) = v_0 \geqq 0$$

is uniformly stable.

Then the system of differential equations (1) *is* (p_0, p)*-stable.*

【Theorem 3.2】 *Assume that*

(B1) $p_0, p \in \Gamma$ *and* p_0 *is finer than* p,

(B2) $V_1 \in C(S(p, \rho), I)$, $V_1(t, x)$ *is locally Lipschitzian in x and weakly* p_0*-decrescent,*

(B3) *for any* $\eta > 0$, *there exists a* $V_{2\eta} \in C(S(p, \rho) \cap S^c(p_0, \eta), I)$,

$V_{2\eta}(t, x)$ *is locally Lipschitzian in x and*

$$a(p(t, x)) \leqq V_{2\eta}(t, x) \leqq b(p_0(t, x)) \text{ on } S(p, \rho) \cap S^c(p_0, \eta),$$

where $a, b \in K$ *and*

$$\dot{V}_1(t, x)_{(1)} + \dot{V}_{2\eta}(t, x)_{(1)} \leqq g(t, V_1(t, x) + V_{2\eta}(t, x)) \text{ on } S(p, \rho) \cap S^c(p_0, \eta),$$

where $g \in C(I \times I, R)$ *with* $g(t, 0) \equiv 0$,

(B4) *the zero solution of the scalar differential equation*

$$u' = g(t, u), \ u(t_0, t_0, u_0) = u_0 \geqq 0$$

is uniformly stable,

(B5) *there exist two functions* $V_3 \in C(S(p, \rho), R)$, $V_4 \in C(S(p, \rho), R)$ *such that* $V_1 = V_3 + V_4$, *where* $V_3(t, x)$ *is p-positive definite and*

$\dot{V}_1(t, x)_{(1)} \leqq -\lambda(t) c(V_3(t, x))$ *on* $S(p, \rho)$, *where* $c \in K$ *and* $\lambda \in C(I, I)$ *is integrally positive, that is,*

$$\int_J \lambda(s)\, ds = \infty \quad \text{whenever } J = \bigcup_{i=1}^{\infty} [\alpha_i, \beta_i],$$

$$\alpha_i < \beta_i < \alpha_{i+1} \text{ and } \beta_i - \alpha_i \geqq \delta > 0,$$

(B6) *for evry function* $y \in C(I, R^n)$, *the function*

$$\int_0^t [\dot{V}_4(s, y(s))]_{\pm} ds$$

is uniformly continuous on I, where $[\bullet]_{\pm}$ *means that either the positive or negative part is considered for all* $s \in I$.

Then the differential system (1) *is* (p_0, p)*-asymptotically stable.*

For proof of these theorems, see [1].

In 2001, we proved the following theorem.

【Theorem 3.3】 *Suppose that*

(C1) $p_0, p \in \Gamma$ *and* p_0 *is finer than* p,

(C2) *for any* $\eta > 0$, *there exists a function* $W_{2\eta} \in C(S(p, \rho) \cap S^c(p_0, \eta), I)$ *such that* $W_{2\eta}(t, x)$ *is locally Lipschitzian in* x *and*

$$a(t, p(t, x)) \leqq W_{2\eta}(t, x) \leqq b(p_0(t, x)) \text{ on } S(p, \rho) \cap S^c(p_0, \eta),$$

where $a(t, r)$ *is continuous in* (t, r), *increasing with respect to* t *for each fixed* r, $a(t, r) > 0$ *for* $r \neq 0$ *and* $a(t, 0) = 0$, $b(r)$ *is continuous, increasing,* $b(r) > 0$ *for* $r > 0$, $b(0) = 0$ *and* $S^c(p_0, \eta)$ *is the complement of* $S(p_0, \eta)$,

(C3) *there exists a function* $V \in C(S(p, \rho), I)$ *satisfying the following properties ;* $V(t, x)$ *is locally Lipschitzian in* x, *weakly* p_0*-decrescent and*

$$\dot{V}(t, x)_{(1)} \leqq g_1(t, V(t, x)),$$

where $g_1 \in C(I \times I, R)$ *and* $g_1(t, 0) \equiv 0$, *and*

$$\dot{V}(t, x)_{(1)} + \dot{W}_{2\eta}(t, x)_{(1)} \leqq g_2(t, V(t, x) + W_{2\eta}(t, x)) \text{ on } S(p, \rho) \cap S^c(p_0, \eta),$$

where $g_2 \in C(I \times I, R)$ *and* $g_2(t, 0) \equiv 0$.

If the zero solution $u = 0$ *of the scalar differential equation*

$$u' = g_1(t, u), \quad u(t_0, t_0, u_0) = u_0, \ u_0 \geqq 0$$

is stable, and the zero solution $v = 0$ *of the scalar differential equation*

$$v' = g_2(t, v), \ v(t_0, t_0, v_0) = v_0, \ v_0 \geqq 0$$

is uniformly stable, then the system of differential equations (1) *is* (p_0, p)*-stable.*

For proof of this theorem, see [2].

4. Main Result

We utilize the ideas of L. Hatvani. (cf [3], [4])

【Theorem 4】 *Suppose that*

(i) $p_0, p \in \Gamma$ *and* p_0 *is finer than* p,

(ii) *for any* $\eta > 0$, *there exists a function* $W_{2\eta} \in C(S(p, \rho) \cap S^c(p_0, \eta), I)$, *where* $S^c(p_0, \eta)$ *is the complement of* $S(p_0, \eta)$, $W_{2\eta}(t, x)$ *is locally Lipschitzian in* x *and*

$$a(t, p(t, x)) \leqq W_{2\eta}(t, x) \leqq b(p_0(t, x)) \text{ on } S(p, \rho) \cap S^c(p_0, \eta),$$

where $a(t, r)$ *is continuous in* (t, r), *increasing with respect to* t *for each fixed* r, $a(t, r) > 0$ *for* $r \neq 0$ *and* $a(t, 0) = 0$, $b(r)$ *is continuous, increasing,* $b(r) > 0$ *for* $r \neq 0$ *and* $b(0) = 0$,

(iii) *there exists a function* $V \in C(S(p, \rho), I)$ *satisfying the following properties*; $V(t, x)$ *is locally Lipschitzian in* x, *weakly* p_0*-decrescent and*

$$\dot{V}(t, x)_{(1)} + \dot{W}_{2\eta}(t, x)_{(1)} \leqq g(t, V(t, x) + W_{2\eta}(t, x)) \text{ on } S(p, \rho) \cap S^c(p_0, \eta),$$

where $g \in C(I \times I, R)$ *and* $g(t, 0) \equiv 0$,

(iv) *there exist two functions* $V_1 \in C(S(p, \rho), R)$, $V_2 \in C(S(p, \rho), R)$ *such that* $V = V_1 + V_2$, $a_1(t, p(t, x)) \leqq V_1(t, x)$, *where* $a_1(t, r)$ *is continuous in* (t, r), *increasing with respect to* r *for each fixed* t, $a_1(t, r) > 0$ *for* $r \neq 0$ *and* $a_1(t, 0) = 0$, *and*

$\dot{V}(t, x)_{(1)} \leqq -\lambda(t) c(V_1(t, x))$ *on* $S(p, \rho)$, *where* $c \in K$ *and* $\lambda \in C(I, I)$ *is integrally positive, that is,*

$$\int_J \lambda(s)\,ds = \infty \quad \textit{whenever } J = \bigcup_{i=1}^{\infty} [\alpha_i, \beta_i],$$

$\alpha_i < \beta_i < \alpha_{i+1}$ *and* $\beta_i - \alpha_i \geqq \delta > 0$,

(v) *for evry function* $y \in C(I, R^n)$, *the function*

$$\int_0^t [\dot{V}_2(s, y(s))]_{\pm}\, ds$$

is uniformly continuous on I, where $[\bullet]_{\pm}$ *means that either the positive or negative part is considered for all* $s \in I$.

If the zero solution $u = 0$ *of the scalar differential equation*

(2) $u' = g(t, u),\ u(t_0, t_0, u_0) = u_0,\ u_0 \geqq 0$

is uniformly stable, then the system of differential equations (1) *is* (p_0, p)*-asymptotically stable.*

Proof. Since (iv) implies that $\dot{V}(t, x)_{(1)} \leqq 0$ on $S(p, \rho)$, all assumptions of Theorem 3.3 are satisfied. Then the system of differential equations (1) is (p_0, p)-stable.

Choosing $\varepsilon = \rho$ and designating *by* $\delta_0 = \delta_0(\rho, t_0) > 0$, it is clear that

(3) $p_0(t_0, x_0) < \delta_0$ implies $p(t, x(t, t_0, x_0)) < \varepsilon$ for all $t \geqq t_0$.

Let $x(t, t_0, x_0)$ be any solution of (1) satisfying (3). Difine the functions $m(t) = V(t, x(t, t_0, x_0))$, $m_1(t) = V_1(t, x(t, t_0, x_0))$ and $m_2(t) = V_2(t, x(t, t_0, x_0))$, so that $m(t) = m_1(t) + m_2(t)$.

Assumptions (iii) and (iv) yield that $m(t)$ is nonincreasing and bounded from below, therefore $\lim_{t\to\infty} m(t) = \sigma < \infty$.

We claim that $\liminf_{t\to\infty} m_1(t) = 0$.

If this were false, there would exist a $\beta > 0$ and a $T > t_0$ such that

(4) $m_1(t) \geqq \beta$ for all $t \geqq T$.

By (iv) and (4), it follows that

$$\dot{m}(t) \leqq -\lambda(t)c(m_1(t)) \leqq -\lambda(t)c(\beta) \ \textit{for all } t \geqq T.$$

Thus, for $J = \bigcup_{i=1}^{\infty} [\alpha_i, \beta_i]$ such that $T < \alpha_i < \beta_i < \alpha_{i+1}$ and $\beta_i - \alpha_i \geqq \beta > 0$, we get

$$\lim_{t\to\infty} m(t) \leqq m(T) - c(\beta)\int_T^{\infty} \lambda(s)\,ds \leqq m(T) - c(\beta)\int_J \lambda(s)\,ds = -\infty,$$

which is a contradiction.

Next, we claim that $\limsup_{t\to\infty} m_1(t) = 0$.

Since $m_1(t) \geqq 0$, suppose that $\limsup_{t\to\infty} m_1(t) > 0$. Then there exists a $\gamma > 0$ such that $\limsup_{t\to\infty} m_1(t) > 3\gamma$.

Since $\lim_{t\to\infty} m(t) = \sigma < \infty$ and $m(t)$ is continuous and nonincreasing, there exists an $M > 0$ such that

(5) $\sigma \leqq m(t) \leqq \sigma + \gamma$ for all $t \geqq t_0 + M$.

For definiteness, suppose that assumption (v) holds with $[\bullet]_+$.

Since $m_1(t)$ is continuous, we can choose a sequence $\{\xi_i\}$ and $\{\pi_i\}$ such that, for $i = 1, 2, \cdots$, $t_0 + M < \xi_i < \pi_i < \xi_{i+1}$,

(6) $m_1(\xi_i) = 3\gamma$, $m_1(\pi_i) = \gamma$ and $\gamma \leqq m_1(t) \leqq 3\gamma$ for all $t \in [\xi_i, \pi_i]$.

From (5) and (6), it is easy to see that

(7) $m(\xi_i) - m_1(\xi_i) \leqq \sigma - 2\gamma$, $m(\pi_i) - m_1(\pi_i) \geqq \sigma - \gamma$.

Since $m_2(t) = m(t) - m_1(t)$, it follows from (7) that

$$0 < \gamma \leqq m_2(\pi_i) - m_2(\xi_i) = \int_{\xi_i}^{\pi_i} [\dot{m}_2(s)]_+ ds,$$

which shows by (v) that there exists a $d > 0$ such that

(8) $\pi_i - \xi_i > d > 0,\ i = 1, 2, \cdots.$

By (6), (8) and (iv), we get

$$\lim_{t\to\infty} m(t) \leqq m(t_0+M) - c(\gamma)\int_{t_0+M}^{\infty} \lambda(s)\,ds \leqq m(t_0+M) - c(\gamma)\int_J \lambda(s)\,ds = -\infty,$$

where $J = \bigcup_{i=1}^{\infty} [\xi_i, \pi_i]$,

which leads to a contradiction.

Then we have $\lim_{t\to\infty} m_1(t) = 0$.

Since properties of a function $V_1(t, x)$, we get $p(t, x(t, t_0, x_0)) \to 0$ as $t \to \infty$.

Thus we can conclude the system of differential equations (1) is (p_0, p)-asymptotically stable.

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