EINSTEIN-WEYL STRUCTURES AND EINSTEIN-HERMITIAN STRUCTURES

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1. INTRODUCTION

Let (M, g, J) be a Hermitian manifold of dimension $2n$, where g is a Riemannian metric and J an almost complex structure. It is known that there is a unique linear connection $\bar{D}$ of M such that $\bar{D}g = 0$ and $\bar{D}J = 0$ and that the torsion tensor T satisfies $T(JX, Y) = T(X, JY)$, where X, Y are any vector fields on M (cf. [7]). The linear connection $\bar{D}$ is called the Hermitian connection. The Hermitian connection $\bar{D}$ is given by

(1) $$2g(\bar{D}_X Y, Z) = 2g(\nabla_X Y, Z) + g(T(X, Y), Z) - g(T(Y, Z), X) + g(T(Z, X), Y),$$

where ∇ is the Levi-Civita connection of g (cf. [5], [8]). Let $\{X_1, ..., X_n, JX_1, ..., JX_n\}$ be an orthonormal basis of T_xM. A Hermitian manifold (M, g, J) is said to have an Einstein-Hermitian structure if

(2) $$Ric^{\bar{D}}(X, Y) := \sum_{i=1}^{n} R^{\bar{D}}(X, JY, X_i, JX_i) = \alpha g(X, Y),$$

where α is a function on M (cf. [4]).

We define the fundamental 2-form Ω by $\Omega(X, Y) = g(X, JY)$. Let ω be the Lee form, i.e., $\omega = \frac{1}{n-1}\delta\Omega \circ J$ and B the Lee vector field dual to ω with respect to g. We define the Weyl connection D of (g, J) by

(3) $$D_X Y = \nabla_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X, Y)B.$$

The Weyl connection D is independent of the choice of g in the conformal class $[g]$. A Hermitian manifold $(M, [g], J, D)$ with the Weyl connection D is called a Hermitian-Weyl manifold. A Weyl manifold $(M, [g], D)$ is said to be Einstein-Weyl if

(4) $$Ric^D(X, Y) + Ric^D(Y, X) = \beta g(X, Y),$$

where β is a function on M and $Ric^D(X, Y) =$ trace of map $Z \mapsto R^D(Z, X)Y$.

It is known that there is a unique metric g, up to a constant, in the conformal structure of a compact Einstein-Weyl manifold with respect to which the corresponding 1-form ω is co-closed. Furthermore, Tod demonstrated that this co-closed 1-form turns out to be the dual of a Killing vector field (cf. [14]).

In [4], Gauduchon and Ivanov proved the following two facts for Hermitian manifold (M, g, J) of dimension 4 :

1. (g, J) is an Einstein-Hermitian structure if and only if $([g], J, D)$ is an Einstein-Weyl structure.
2. If (M, g, J) is a compact Einstein-Hermitian manifold, then :

(i) either (M, g, J) is Einstein-Kähler, or
(ii) (M, g) is locally isometric to $\mathbf{R} \times S^3$.

In this paper, we consider a Hermitian manifold (M, g, J) of dimension $2n \geq 4$. And we prove the following results.

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Theorem 1. *Let $(M, [g], J, D)$ be a Hermitian-Weyl manifold of dimension $2n \geq 4$ and satisfies $DJ = 0$. Then (g, J) is an Einstein-Hermitian structure if and only if $([g], J, D)$ is an Einstein-Weyl structure.*

Theorem 2. *Let $(M, [g], J, D)$ be a compact Hermitian-Weyl manifold of dimension $2n \geq 4$ and satisfies $DJ = 0$. If (g, J) is an Einstein-Hermitian structure, then :*
(i) *either (M, g, J) is Einstein-Kähler, or*
(ii) *M has the first Betti number $b_1(M) = 1$ and the universal covering manifold of (M, g) is isometric to $\mathbf{R}\times N^{2n-1}$, where N^{2n-1} is a simply connected Ricci positive Einstein manifold, moreover, (M, g, J) is a generalized Hopf manifold and N^{2n-1} is a Sasakian manifold.*

Remark. If $(M, [g], J, D)$ is a Hermitian-Weyl manifold of dimension 4, then we have $d\Omega = \omega \wedge \Omega$ and so $DJ = 0$ but ω is not necessarily closed (cf. [17]).

2. PRELIMINARIES

Let $(M, [g], J, D)$ be a Hermitian-Weyl manifold. From (1) and (3), we obtain

$$2g(\bar{D}_X Y, Z) = 2g(D_X Y, Z) + \omega(X)g(Y, Z) + \omega(Y)g(X, Z) - \omega(Z)g(X, Y) + g(T(X, Y), Z) - g(T(Y, Z), X) + g(T(Z, X), Y). \tag{5}$$

Since the almost complex structure J has no torsion, using $\bar{D}J = 0$ and $T(JX, Y) = T(X, JY)$, we obtain $T(JX, Y) = JT(X, Y)$.

Thus, from (5), we have

$$(D_X \Omega)(Y, Z) = (\bar{D}_X \Omega)(Y, Z) + \omega(X)\Omega(Y, Z) - \frac{1}{2}\omega(Y)g(JX, Z) - \frac{1}{2}\omega(JZ)g(X, Y) - \frac{1}{2}\omega(Z)g(X, JY) + \frac{1}{2}\omega(JY)g(X, Z) + g(T(Y, Z), JX). \tag{6}$$

Lemma 1. *On a Hermitian-Weyl manifold $(M, [g], J, D)$, the following conditions are equivalent :*
(i) $DJ = 0$,
(ii) $d\Omega = \omega \wedge \Omega$,
(iii) $D_X \Omega = \omega(X)\Omega$,
(iv) $\bar{D}_X Y = D_X Y + \frac{1}{2}\omega(X) Y - \frac{1}{2}\omega(JX)JY$.

Proof. By a Theorem of Vaisman [16], we have the equivalence of (i), (ii) and (iii).

Now, we prove the equivalence of (iii) and (iv).

We assume that $D_X \Omega = \omega(X)\Omega$. Since $\bar{D}g = 0$ and $\bar{D}J = 0$, we obtain $\bar{D}_X \Omega = 0$. Hence, from (6) we have

$$g(T(Y, Z), X) = \frac{1}{2}\omega(Y)g(X, Z) - \frac{1}{2}\omega(JZ)g(JX, Y) - \frac{1}{2}\omega(Z)g(X, Y) + \frac{1}{2}\omega(JY)g(JX, Z). \tag{7}$$

By using (5), from (7) we obtain

$$\bar{D}_X Y = D_X Y + \frac{1}{2}\omega(X) Y - \frac{1}{2}\omega(JX)JY. \tag{8}$$

Conversely, assume the condition (8). Since $T(X, Y) = \bar{D}_X Y - \bar{D}_Y X - [X, Y]$ and D is torsion-free, we obtain (7). From (6), we have $D_X \Omega = \omega(X)\Omega$. □

The curvature tensor R of the Levi-Civita connection ∇ defined by $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$. We set $R(V, Z, X, Y) = g(R(X, Y)Z, V)$. Let R^D and $R^{\bar{D}}$ be the curvature tensors of D and $\bar{D}$ respectively. By a simple calculation, from (1) and (3), we have

Lemma 2. (i) $R^D(V, Z, X, Y) = -R^D(V, Z, Y, X)$,
(ii) $R^D(V, Z, X, Y) + R^D(Z, V, X, Y) = -2d\omega(X, Y)g(Z, V)$,
(iii) $R^D(V, Z, X, Y) + R^D(V, X, Y, Z) + R^D(V, Y, Z, X) = 0$,
(iv) $R^{\bar{D}}(V, Z, X, Y) = -R^{\bar{D}}(V, Z, Y, X)$,
(v) $R^{\bar{D}}(V, Z, X, Y) = -R^{\bar{D}}(Z, V, X, Y)$,
(iv) $R^{\bar{D}}(JV, JZ, X, Y) = R^{\bar{D}}(V, Z, JX, JY) = R^{\bar{D}}(V, Z, X, Y)$,
where $2d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$.

3. Proofs

Proof of Theorem 1. From (iv) of Lemma 1, we obtain

(9) $$R^{\bar{D}}(V, Z, X, Y) = R^D(V, Z, X, Y) + g(V, Z)\, d\omega(X, Y) - \Omega(V, Z)\, d\theta(X, Y),$$

where $\theta = \omega \circ J$ (cf. [17]). Since $DJ = 0$, we have $R^D(JV, JZ, X, Y) = R^D(V, Z, X, Y)$. Using Lemma 2 and (9), we obtain

$$\begin{aligned} Ric^D(X, Y) &= \sum_{i=1}^{n} (R^D(X_i, Y, X_i, X) + R^D(JX_i, Y, JX_i, X)) \\ &= \sum_{i=1}^{n} (R^D(JY, JX_i, X, X_i) + R^D(JY, X_i, JX, X)) + 2d\omega(X, Y) \\ &= -\sum_{i=1}^{n} R^D(JY, X, X_i, JX_i) + 2d\omega(X, Y) \\ &= \sum_{i=1}^{n} (R^{\bar{D}}(X, JY, X_i, JX_i) + d\omega(X_i, JX_i) g(X, JY) + d\theta(X_i, JX_i)\Omega(X, JY)) \\ &\quad + 2d\omega(X, Y) \\ &= Ric^{\bar{D}}(X, Y) + \sum_{i=1}^{n} d\omega(X_i, JX_i) g(X, JY) - \sum_{i=1}^{n} d\theta(X_i, JX_i) g(X, Y) + 2d\omega(X, Y). \end{aligned}$$

Thus we obtain

(10) $$Ric^D(X, Y) + Ric^D(Y, X) = 2Ric^{\bar{D}}(X, Y) - 2\sum_{i=1}^{n} d\theta(X_i, JX_i) g(X, Y).$$

Therefore (g, J) is an Einstein-Hermitian structure if and only if $([g], J, D)$ is an Einstein-Weyl strucuture.

Proof of Theorem 2. From Theorem 1, $(M, [g], J, D)$ is a compact Einstein-Weyl manifold, thus we can choose a Riemannian metric g such that $\delta\omega = 0$ and Lee vector field B is Killing. In the case where dimension of M is 4, this Theorem has been gave by Gauduchon and Ivanov (cf. [4]). We assume that dim $M \geq 6$. From (ii) of Lemma 1 and dim $M \geq 6$, ω is closed. So ω is harmonic and parallel with respect to the Levi-Civita connection ∇ of g. Since $\nabla\omega = 0$, $|\omega|$ is constant. Since $(M, [g], J, D)$ is an Einstein-Weyl manifold with the Killing vector field B and $|\omega|$ is constant, s^D is constant, where s^D is the scalar curvature of the Weyl connection D (cf. [3], [6]).

Since $(M, [g], J, D)$ is an Einstein-Weyl manifold from (3), we have

(11) $$Ric(X, Y) = \frac{1}{2n} s^D g(X, Y) + \frac{n-1}{2}(|\omega|^2 g(X, Y) - \omega(X)\omega(Y)),$$

where Ric is the Ricci curvature of the Levi-Civita connection ∇ (cf. [12]). Thus we obtain $Ric(\omega) =$

$\frac{1}{2n} s^D\omega$. Since the dual of ω is Killing, we have $\nabla^*\nabla\omega = Ric(\omega)$ (cf. [1] p. 41). So we have $\nabla^*\nabla\omega = \frac{1}{2n} s^D\omega$. We integrate over M the scalar product of $\nabla^*\nabla\omega$ with ω. Then we obtain $\int_M |\nabla\omega|^2 dV_g = \frac{1}{2n} s^D |\omega|^2 \int_M dV_g$, where dV_g denotes the volume element with respect to g. Since ω is parallel, we obtain $\omega = 0$ or $s^D = 0$.

In the case where $\omega = 0$, since $\nabla J = DJ = 0$, (M, g, J) is an Einstein-Kähler manifold.

In the case where $s^D = 0$, from (11), for any tangent vector field X orthogonal to B, we obtain

$$Ric(B, B) = 0, \quad Ric(B, X) = 0$$

and

$$Ric(X, X) = \frac{n-1}{2} |\omega|^2 g(X, X).$$

By the splitting theorem on nonnegative Ricci curvature (cf. [1], [2]), the universal covering manifold of (M, g) is isometric to $\mathbf{R} \times N^{2n-1}$, where N^{2n-1} is a simply connected Ricci positive Einstein manifold.

Since ω is harmonic and the Ricci curvature is nonnegative, using the Weizenböck formula, we obtain $b_1(M) = 1$ (cf. [12], [6]).

Since $d\Omega = \omega \wedge \Omega$ and ω is parallel, (M, g, J) is a generalized Hopf manifold (in terminology of Vaisman). Since N^{2n-1} is orthogonal to the Lee vector field B, N^{2n-1} is the universal covering space of a leaf of the canonical foliation defined by $\omega = 0$. From a Theorem of Vaisman [15], N^{2n-1} is a Sasakian manifold.

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