On Partially Integral Stability

Masamichi ASO, Shoichi SEINO, and Miki KUDO

(Received on 30, November, 1998)

1. Introduction

Ivo Vrkoc [1] introduced the concept of integral stability based on the idea of considering perturbations which can be great in certain moments but are small in the mean. Many authors have discussed the integral stability. (cf. [1], [2], [4], [11], [12], [13], [14], [15]).

The concept for integral stability with respect to the variable, which is introduced in the paper [1], is generalized in [2] in this way.

As is well known, Liapunov's second method has its origin in three simple theorems that from the core of what he himself called the second method for dealing with questions of stability. It is widely recognized as an indispensable tool not only in the theory of stability but also in studying many other qualitative properties of solutions of differential equations. The main characteristic of this method is the introduction of a function, namely the Liapunov function, which defines a generalized distance from the origin of the motion space. Liapunov's second method is a very useful and powerful instrument in discussing the stability of the system of differential equations.

Its power and usefulness lie in the fact that the decision is made by investigating the differential equations itself and not by finding solutions of the differential equations. However, it is great difficult to find the Liapunov function satisfying certain conditions. Therefore, it is important to obtain a weak sufficient condition for a stability theorem.

In many applications, we need to see the qualities not of the whole solution but of the partial.

In this paper, by using the Liapunov's second method and the comparison principle, we will state extension of the sufficient conditions for the partially integral stability.

2. Notations and Definitions

First, we summarize some basic notations and definitions we will need later on.

Let I denote the interval $0 \le t < \infty$, R^n denote Euclidean n-space. For $x \in R^n$, $y \in R^m$ and $H > 0$, $S(H) = \{(x, y) \in R^{n+m};\ \|x\| < H,\ \|y\| < \infty\}$ and $Q(H) = \{(x, y) \in R^{n+m};\ \|x\| + \|y\| < H\}$, where $\|x\|$ is any norm of x. We shall denote by $C(I \times R^n \times R^m, R^k)$ the set of all continuous functions defined on $I \times R^n \times R^m$ with valued in R^k.

We consider a system of differential equations

$$(1)\qquad \begin{cases} \dfrac{dx}{dt} = f(t, x, y),\ f(t, 0, 0) \equiv 0, \text{ where } f(t, x, y) \in C(I \times R^n \times R^m, R^n) \\[2ex] \dfrac{dy}{dt} = g(t, x, y),\ g(t, 0, 0) \equiv 0, \text{ where } g(t, x, y) \in C(I \times R^n \times R^m, R^m), \end{cases}$$

and its perturbed system

$$(2)\qquad \begin{cases} \dfrac{dx}{dt} = f(t, x, y) + F(t, x, y), \text{ where } F(t, x, y) \in C(I \times R^n \times R^m, R^n) \\[2ex] \dfrac{dy}{dt} = g(t, x, y) + G(t, x, y), \text{ where } G(t, x, y) \in C(I \times R^n \times R^m, R^m). \end{cases}$$

We assume that $f(t,x,y)$, $g(t,x,y)$, $F(t,x,y)$ and $G(t,x,y)$ are smooth enough to existence, uniqueness and continuous dependence of the solutions of the initial value problem.

We introduce the following definitions.

Definition 1 *Corresponding to a scalar function $V(t,x,y)$ defined on an open set, we define the function*

$$\dot{V}_{(1)}(t,x,y) = \overline{\lim_{h\to 0+}}\frac{1}{h}\{V(t+h, x+hf(t,x,y), y+hg(t,x,y)) - V(t,x,y)\}.$$

In case $V(t,x,y)$ has continuous partial derivatives of the first, it is evident that

$$\dot{V}_{(1)}(t,x,y) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\bullet f(t,x,y) + \frac{\partial V}{\partial y}\bullet g(t,x,y),$$

where "$\bullet$" denotes a scalar product.

Next, let us define the notion of the partially integral stability.

Definition 2 *The zero solution of the system (1) is said to be partially integrally stable with respect to x if for any $\epsilon > 0$ and any $t_0 \geq 0$ there exist $\delta_1(t_0,\epsilon) > 0$, $\delta_2(t_0,\epsilon) > 0$ such that $\|x_0\| + \|y_0\| < \delta_1(t_0,\epsilon)$ and $\int_{t_0}^{\infty}\sup_{\|x\|<\epsilon}\{\|F(t,x,y)\| + \|G(t,x,y)\|\}dt < \delta_2(t_0,\epsilon)$ implies $\|x(t,t_0,x_0,y_0)\| < \epsilon$ for all $t \geq t_0$, where $(x(t,t_0,x_0,y_0), y(t,t_0,x_0,y_0))$ denotes a solution of the system (2) satisfying a initial condition $(x(t_0,t_0,x_0,y_0), y(t_0,t_0,x_0,y_0)) = (x_0,y_0)$.*

Definition 3 *The zero solution of the system (1) is said to be uniformly partially integrally stable with respect to x if δ_1 and δ_2 in Definition 2 are independent of t_0.*

We consider the scalar differential equation

(3) $$\frac{du}{dt} = g(t,u),\ u(t_0) = u_0,$$

where $g \in C(I\times I, R)$ and $g(t,0) \equiv 0$, and its perturbed differential equation

(4) $$\frac{du}{dt} = g(t,u) + \phi(t),$$

where $\phi \in C(I,R)$.

Definition 4 *The zero solution $u = 0$ of (3) is said to be integrally stable, if for any $\epsilon > 0$ and any $t_0 \geq 0$ there exist $\delta_1(t_0,\epsilon) > 0$, $\delta_2(t_0,\varepsilon) > 0$ such that, for any solution $u(t,t_0,u_0)$ of the perturbed differential equation (4), $u_0 < \delta_1(t_0,\varepsilon)$ and*

$$\int_{t_0}^{\infty}\phi(t)\,dt < \delta_2(t_0,\varepsilon)$$

implies $u(t,t_0,u_0) < \epsilon$ for all $t \geq t_0$.

3. Preliminary results

In 1994, I.K.Russinov proved the next theorem.

Theorem 1 *Assume there exist vector-functions $V_1(t,x,y)$ and $V_{2,\eta}(t,x,y)$ satisfying the following statements :*

(i) $V_1(t,x,y) \in C(R^+\times S(\rho), R^+)$, $V_1(t,x,y)$ is locally Lipschitzian with respect to (x,y), $V_1(t,0,y) \equiv 0$ and

$$D^+V_1(t,x,y) \leq g_1(t, V_1(t,x,y)),\ (t,x,y) \in R^+\times S(\rho),$$

where $g_1(t, u) \in C(R^+ \times R^+, R)$ *and* $g_1(t, 0) \equiv 0$;

(ii) for each $\eta > 0$ *there exists* $V_{2,\eta}(t, x, y) \in C(R^+ \times S(\rho) \cap Q^c(\eta), R^+)$,

$$a(\|x\|) \leq V_{2,\eta}(t, x, y) \leq b(\|x\| + \|y\|)$$

for any $(t, x, y) \in R^+ \times S(\rho) \cap Q^c(\eta)$, *where* $a, b \in C((0, \rho), R^+)$, $a(u)$ *and* $b(u)$ *are increasing with respect to* u, $\lim_{u \to 0} a(u) = 0$ *and*

$$D^+ V_1(t, x, y) + D^+ V_{2,\eta}(t, x, y)$$
$$\leq g_2(t, V_1(t, x, y) + V_{2,\eta}(t, x, y), x)$$

for all $(t, x, y) \in R^+ \times S(\rho) \cap Q^c(\eta)$, *where* $g_2(t, u, v) \in C(R^+ \times R^+ \times S(\rho), R)$, $g_2(t, 0, 0) \equiv 0$;

(iii) the zero solution of the scalar equation

$$\frac{du}{dt} = g_1(t, u), \ u(t_0) \geq 0 \tag{5}$$

is integrally stable and the zero solution $(x = 0, y = 0, v = 0)$ *of the system (1) and the equation*

$$\frac{dv}{dt} = g_2(t, v, x) \tag{6}$$

is uniformly integrally v-stable.

Then the zero solution of system (1) is integrally x-stable.

For the proof of this theorem, see [4].

Theorem 2 *Suppose that there exist functions* $V(t, x, y) \in C(I \times S_H \times R^m, I)$ *and* $h(t) \in C(I, I)$, *which satisfies the following conditions :*

(i) $a(t, \|x\|) \leq V(t, x, y)$, $V(t, 0, 0) = 0$, *where* $a(t, r)$ *is continuous in* (t, r), *nondecreasing in* r *for each* t, *nondecreasing in* t *for each* r, $a(t, r) > 0$ *for all* $r \neq 0$ *and* $a(t, 0) = 0$,

(ii) $|V(t, x_1, y_1) - V(t, x_2, y_2)| \leq M\{\|x_1 - x_2\| + \|y_1 - y_2\|\}$, $M > 0$,

(iii) $\dot{V}_{(1)}(t, x, y) \leq h(t) V(t, x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0))$ *with* $h(t) \geq 0$, $\int_0^\infty h(t)\,dt < \infty$, *where* $(x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0))$ *is any solution of the system (1), then the zero solution of the system (1) is partially uniformly integrally stable with respect to x.*

For the proof of this theorem, see [11].

Theorem 3 *Assume that there exist functions* $V(t, x, y)$ *and* $a(t, r)$ *satisfying the following properties :*

(i) $V \in C(I \times R^n \times R^m, I)$, $V(t, x, y)$ *is Lipschitzian in* (x, y) *for a constant* $M > 0$ *and* $V(t, 0, 0) = 0$,

(ii) $a(t, \|x\|) \leq V(t, x, y)$, *where* $a \in C(I \times I, I)$ *and monotone increasing with respect to* t *for each fixed* r *and* $a(t, r) > 0$ *for* $r \neq 0$,

(iii) $\dot{V}_{(1)}(t, x, y) \leq g(t, V(t, x, y))$ *for* $(t, x, y) \in I \times R^n \times R^m$.

$$\frac{du}{dt} = g(t, u), \ u(t_0) = u_0 \geq 0. \tag{7}$$

Then the equi-integral stability of the null solution $u = 0$ *of (7) implies that the trivial solution* $z = 0$ *of (1) is partially integrally stable with respect to x.*

For the proof of this theorem, see [12].

4. Main result

Theorem 4 *Suppose that there exists a vector function* $(V(t, x, y), W(t, x, y))$ *which satisfies the following conditions :*

(i) $V(t,x,y) \in C(I\times S(H), \mathrm{I})$ *is locally Lipschitzian with respect to* (x,y), $V(t,0,0)\equiv 0$ *and*

$$\dot{V}_{(1)}(t,x,y) \le g_1(t, V(t,x,y)),\ (t,x,y) \in I\times S(H),$$

where $g_1(t,u) \in C(I\times I, R)$ *and* $g_1(t,0)\equiv 0$,

(ii) for any $\eta > 0$, $W(t,x,y) \in C(I\times S(H)\cap Q^c(\eta), I)$ *is locally Lipschitzian with respect to* (x,y),

$$a(t, \|x\|) \le W(t,x,y) \le b(\|x\|+\|y\|)$$

for any $(t,x,y) \in I\times S(H)\cap Q^c(\eta)$, *where* $a(t,r)$ *is continuous in* (t,r), *nondecreasing in* r *for each* t, *nondecreasing in* t *for each* r, $a(t,r) > 0$ *for any* $r > 0$ *and* $a(t,0)\equiv 0$, $b(r)$ *is continuous and nondecreasing in* r, $b(0)\equiv 0$ *and*

$$\dot{V}_{(1)}(t,x,y) + \dot{W}_{(1)}(t,x,y) \le g_2(t, V(t,x,y)+W(t,x,y), x),$$

where $g_2(t,v,x) \in C(I\times I\times S(H), R)$ *and* $g_2(t,0,0)\equiv 0$,

(iii) the zero solution $u=0$ *of the scalar differential equation*

$$\frac{du}{dt} = g_1(t,u),\ u(t_0) = u_0 \ge 0 \tag{8}$$

is integrally stable and the zero solution $(x=0,\ y=0,\ v=0)$ *of the system (1) and the equation*

$$\frac{dv}{dt} = g_2(t,v,x) \tag{9}$$

is uniformly partially integrally stable with respect to v.

Then the zero solution of system (1) is integrally stable with respect to x.

Proof. Since functions $V(t,x,y)$ and $W(t,x,y)$ are locally Lipschitzian with respect to (x,y), there exist $M_1 > 0$ and $M_2 > 0$ such that

$$\|V(t,x_1,y_1) - V(t,x_2,y_2)\| \le M_1(\|x_1-x_2\|+\|y_1-y_2\|) \tag{10}$$

$$\|W(t,x_1,y_1) - W(t,x_2,y_2)\| \le M_2(\|x_1-x_2\|+\|y_1-y_2\|). \tag{11}$$

Let $(x(t,t_0,x_0,y_0),\ y(t,t_0,x_0,y_0))$ is the solution of the system (2) satisfying the initial condition $(x(t_0,t_0,x_0,y_0),\ y(t_0,t_0,x_0,y_0)) = (x_0,y_0)$. From condition (ii) of the theorem and inequalities (10) and (11), we obtain

$$\begin{aligned} &\dot{V}_{(2)}(t,x,y) + \dot{W}_{(2)}(t,x,y) \\ &\le \dot{V}_{(1)}(t,x,y) + \dot{W}_{(1)}(t,x,y) + (M_1+M_2)(\|F(t,x,y)\|+\|G(t,x,y)\|) \\ &\le g_2(t, V(t,x,y)+W(t,x,y), x) + (M_1+M_2)(\|F(t,x,y)\|+\|G(t,x,y)\|). \end{aligned} \tag{12}$$

Let now it be any given $\varepsilon > 0 (0 < \varepsilon < H)$. Because the zere solution $x=0$, $y=0$, $v=0$ of the system (1) and equation (9) is uniformly integrally stable with respect to v, for any $a(0,\varepsilon) > 0$ and any $t_0 \in I$ there exist $\delta_1(\varepsilon) > 0$ and $\delta_2(\varepsilon) > 0$ such that for each solution $w(t,t_0,w_0)$ of the equation

$$\frac{dw}{dt} = g_2(t,w,x) + (M_1+M_2)(\|F(t,x,y)\|+\|G(t,x,y)\|), \tag{13}$$

$\|w_0\| < \delta_1$ and

$$(M_1+M_2)\int_{t_0}^{\infty} \sup_{(x,y)\in S(\varepsilon)} (\|F(t,x,y)\|+\|G(t,x,y)\|)\,dt \le \delta_2(\varepsilon)$$

implies

$$w(t,t_0,w_0) < a(0,\varepsilon) \ \textit{for all}\ t \ge t_0. \tag{14}$$

From the assumption of the function $b(r)$, we have that for each $\frac{1}{2}\delta_1(\varepsilon) > 0$ there exists $\delta_3(\delta_1(\varepsilon)) = \delta_3(\varepsilon) > 0$ such that $\|x\|+\|y\| < \delta_3(\varepsilon)$ implies

$$W(t,x,y) \le b(\|x\|+\|y\|) < b(\delta_3(\varepsilon)) < \frac{1}{2}\delta_1(\varepsilon). \tag{15}$$

Since the integral stability of the zero solution $u=0$ of (8), for $\frac{1}{2}\delta_1(\varepsilon) > 0$ and any $t_0 \in I$ there exist $\delta_4(t_0, \varepsilon) > 0$ and $\delta_5(t_0, \varepsilon) > 0$ such that for the solution $z(t, t_0, z_0)$ of the equation

(16) $$\frac{dz}{dt} = g_1(t, z) + M_1(\|F(t, x, y)\| + \|G(t, x, y)\|),$$

we have $z(t, t_0, z_0) < \frac{1}{2}\delta_1(\varepsilon)$ for all $t \geq t_0$ as $\|z_0\| < \delta_4(t_0, \varepsilon)$ and

$$M_1 \int_{t_0}^{\infty} \sup_{(x,y) \in S(\varepsilon)} (\|F(t, x, y)\| + \|G(t, x, y)\|)\ dt \leq \delta_5(t_0, \varepsilon).$$

We put $z_0 = V(t_0, x_0, y_0)$. Because $V(t, x, y)$ is continuous and $V(t, 0, 0) \equiv 0$, for given $\delta_4(t_0, \varepsilon) > 0$ and any $t_0 \in I$, there exists $\delta_6(t_0, \varepsilon) > 0$ such that $\|x_0\| + \|y_0\| < \delta_6(t_0, \varepsilon)$ implies $V(t_0, x_0, y_0) < \delta_4(t_0, \varepsilon)$.

Let $\delta_7(t_0, \varepsilon) \equiv \min\{\delta_3(\varepsilon), \delta_6(t_0, \varepsilon)\}$. We shall prove that for each solution $(x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0))$ of the system (2)
$\|x_0\| + \|y_0\| < \delta_7(t_0, \varepsilon)$ and

$$\int_{t_0}^{\infty} \sup_{(x,y) \in S(\varepsilon)} \{\|F(t, x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0))\| + \|G(t, x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0))\|\} dt < \delta_8(t_0, \varepsilon)$$

implies $\|x(t, t_0, x_0, y_0)\| < \varepsilon$ for all $t \geq t_0$, where

$$\delta_8(t_0, \varepsilon) \equiv \min\left\{\frac{\delta_2(\varepsilon)}{M_1 + M_2}, \frac{\delta_5(t_0, \varepsilon)}{M_1}\right\}.$$

Let this is not true, then there exist t_1 and t_2 such that for the solution $(x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0))$ of the system (2),

$$\|x(t_1, t_0, x_0, y_0)\| + \|y(t_1, t_0, x_0, y_0)\| = \delta_3(\varepsilon), \ \|x(t_2, t_0, x_0, y_0)\| = \varepsilon$$

and $\delta_3(\varepsilon) \leq \|x(t, t_0, x_0, y_0)\| + \|y(t, t_0, x_0, y_0)\| \leq \varepsilon$ for all $t \in [t_1, t_2]$, where $\|x_0\| + \|y_0\| < \delta_7(t_0, \varepsilon)$. We choose $\eta = \delta_3(\varepsilon) > 0$ such that there exists $W(t, x, y)$ which satisfies the condition (ii) of the theorem. For any $t \in [t_1, t_2]$, from (12), we obtain

$$\begin{aligned} &\dot{V}_{(2)}(t, x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0)) + \dot{W}_{(2)}(t, x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0)) \\ &\quad \leq g_2(t, V(t, x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0)) + W(t, x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0)), x(t, t_0, x_0, y_0)) \\ &\quad + (M_1 + M_2)\{\|F(t, x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0))\| + \|G(t, x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0))\|\}. \end{aligned}$$

The comparison principle implies the following inequality.

$$\begin{aligned} V(t_2, x(t_2, t_0, x_0, y_0), y(t_2, t_0, x_0, y_0)) &+ W(t_2, x(t_2, t_0, x_0, y_0), y(t_2, t_0, x_0, y_0)) \\ &\leq \bar{w}(t_2, t_1, V(t_1, x(t_1, t_0, x_0, y_0), y(t_1, t_0, x_0, y_0)) \\ &\quad + W(t_1, x(t_1, t_0, x_0, y_0), y(t_1, t_0, x_0, y_0)), \end{aligned}$$

where $\bar{w}(t, t_0, w_0)$ is the maximal solution of the equation (13) such that

$$\begin{aligned} \bar{w}_0 &= w(t_1, t_1, w_0) \\ &= V(t_1, x(t_1, t_0, x_0, y_0), y(t_1, t_0, x_0, y_0)) + W(t_1, x(t_1, t_0, x_0, y_0), y(t_1, t_0, x_0, y_0)). \end{aligned}$$

From the condition (i) and the comparison principle, we have

$$V(t, x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0)) \leq \bar{z}(t, t_0, z_0)$$

for all $t \geq t_0$, where $\bar{z}(t, t_0, z_0)$ is the maximal solution of the equation (16). Then we have

$$V(t_1, x(t_1, t_0, x_0, y_0), y(t_1, t_0, x_0, y_0)) \leq \bar{z}(t_1, t_0, z_0) = \bar{z}(t_1, t_0, V(t_0, x_0, y_0)).$$

From (16) and the fact $V(t_0, x_0, y_0) < \delta_4(t_0, \varepsilon)$,

$$V(t_1, x(t_1, t_0, x_0, y_0), y(t_1, t_0, x_0, y_0)) < \frac{1}{2}\delta_1(\varepsilon),$$

and the assumption for the function $W(t, x, y)$, we have

$$\begin{aligned} W(t_1, x(t_1, t_0, x_0, y_0), y(t_1, t_0, x_0, y_0)) &\leq b(\|x(t_1, t_0, x_0, y_0)\| + \|y(t_1, t_0, x_0, y_0)\|) \\ &= b(\varepsilon) < \frac{1}{2}\delta_1(\varepsilon). \end{aligned}$$

Therefore

$$|V(t_1, x(t_1, t_0, x_0, y_0), y(t_1, t_0, x_0, y_0)) + W(t_1, x(t_1, t_0, x_0, y_0), y(t_1, t_0, x_0, y_0))| < \delta_1(\varepsilon)$$

implies $\bar{w}(t_2, t_1, \bar{w}_0) < a(0, \varepsilon)$. By the assumption $\|x(t_2, t_0, x_0, y_0)\| = \varepsilon$, it follows that

$$\begin{aligned} a(0, \varepsilon) = a(0, \|x(t_2, t_0, x_0, y_0)\|) &\leq W(t_2, x(t_2, t_0, x_0, y_0), y(t_2, t_0, x_0, y_0)) \\ &\leq V(t_2, x(t_2, t_0, x_0, y_0), y(t_2, t_0, x_0, y_0)) + W(t_2, x(t_2, t_0, x_0, y_0), y(t_2, t_0, x_0, y_0)) \\ &\leq \bar{w}(t_2, t_1, \bar{w}_0) < a(0, \varepsilon), \end{aligned}$$

which contradicts. Thus, the proof is complete.

References

[1] I. Vrkoc: Integral Stability, *Czechoslovak Math. J.,* Vol. **9**, 1959, 71-129.

[2] V.V. Rumyantsev, A.S. Oziraner: Stability and Stabilization with respect to a part of the Variables-M, *Nauka*, 1987.

[3] V.V. Rumiantsev: On the Stability of a Motion in a Part Variables, *Vestnik Moskov Univ. Ser. 1, Math. Meh.*, **4**, 1957, 9-16.

[4] I.K. Russinov: Integral Stability with respect to a Part of the Variables and Perturbing Lyapunov Functions, *Plovdiv University "Paissii Hilendarrski" Bulgaria, Scientific Works, Vol.* **31**, Book 3, 1994, 83-87.

[5] T. Yosizawa: Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions, *Supringer-Verlag*, 1975.

[6] T. Yosizawa: Stability Theory by Liapunov's Second Method, *The Math. Soci. of Japan*, 1966.

[7] A. Haranay: Differential Equation, *Academic Press*, 1966.

[8] V. Lakshmikantham and S. Leela: Differential and Integral Inequalities, Vol. 1, Ordinary Differential Equations, *Academic Press*, 1969.

[9] N. Rouche, P. Habets and M. Laloy: Stability Theory by Liapunov's Direct Method, *Springer-Verlag*, 1977.

[10] C. Corduneanu: Sur la Stabilite Partielle, *Rev. Math. Pres Appl.*, 1964, 229-236.

[11] S. Seino: On the Partially Uniformly Integral Stability of Solutions of Ordinary Differential Equations, *Reseach Reports of Akita National College of Technology, Vol.* **25**, 1990, 78-83.

[12] M. Kudo: Partially Integral Stability Theorems by the Comparison Principle, *Reseach Reports of Akita National College of Technology, Vol.* **25**, 1990, 84-89.

[13] S. Seino: On Uniformly Integral Stability of Solutions of Ordinary Differential Equations, *Reseach Reports of Akita National College of Technology, Vol.* **23**, 1988, 60-66.

[14] M. Kudo: On the Integral Stability and the Uniformly Integral Stability of Nonlinear Differential Equations by Using Comparison Principle, *Reseach Reports of Akita National College of Technology, Vol.* **23**, 1988, 67-72.

[15] M. Aso, M. Kudo and S. Seino: Asymptotically Integral Stability Theorem by the Comparison Principle, *Reseach Reports of Akita National College of Technology, Vol.* **23**, 1988, 73-76.

[16] S. Seino, M. Kudo and M. Aso: On Partial Stability of Solutions of a System of Ordinary Differential Equations, *Reseach Reports of Akita National College of Technology, Vol.* **16**, 1981, 122-125.

[17] S. Seino, M. Kudo and M. Aso: Partial Stability Theorems by the Comparison Principle, *Reseach Reports of Akita National College of Technology, Vol.* **17**, 1982, 95-99.

[18] S. Seino, M. Kudo and M. Aso: Partial Stability Theorems by Liapunov's Second Method, *Reseach Reports of Akita National College of Technology, Vol.* **18**, 1983, 132-135.

[19] M. Kudo, S. Seino and M. Aso: On the Partially Total Stability and the Partially Total Boundedness of a system of Ordinary Differential Equations, *Reseach Reports of Akita National*

College of Technology, Vol. **20**, 1985, 105-109.
[20] M. Kudo, M. Aso and S. Seino : On Partial L^p-Stability, *Reseach Reports of Akita National College of Technology, Vol*. **24**, 1989, 71-75.