On Stability of Difference Equations with Finite Delay

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1. Introduction

Liapunov's second method is widely recognized as an indispensable tool in studying not only the theory of stability but also many other qualitative properties of solutions of differential equations.

In difference equations, stability and boundedness are very important problems. To apply Liapunov's second method to difference equations, one must actually use a Liapunov functional and a Liapunov function. By using Liapunov functional and Liapunov function, one can extend to difference equations most of the well-known results for functional differential equations.

In this paper, we study stability theorem which use a Liapunov functional and a Liapunov function satisfying weaker conditions.

2. Definitions and Notations

Let Z denote the set of all integers and Z^+ denote the set of non negative integers. Let R^+ denote the interval $[0, \infty)$ and R^k denote Euclidean k-space. For $x \in R^k$, $|x|$ is the norm of x. We consider the difference systems with finite delay,

$$x(n+1) = F(n, x(n), x(n-\tau_1(n)), x(n-\tau_2(n)), \ldots, x(n-\tau_m(n))), \tag{1}$$

where $\tau_i \in Z^+$ is defined on Z^+, $0 \leq \tau_i(n) \leq r\,(i = 1, 2, \ldots, m)$, for some positive integer r. We suppose

$$F(n, 0, 0, \ldots, 0) \equiv 0$$

for all $n \in Z^+$ so that (1) always has the zero solution $x \equiv 0$ for $n \in Z^+$. Let $I_r = \{-r, -r+1, \ldots, -1, 0\}$. For a given function $\varphi \in R^k$ which is defined on I_r, $\|\varphi\| = sup\,\{|\varphi(j)| : j \in I_r\}$. For any $H > 0$, S_H denote the set of φ such that $\|\varphi\| < H$. For any function $x(m)$ defined on $m \geq -r$, for any fixed $n \in Z^+$, the symbol x_n denote the restriction of $x(m)$ to $n-r \leq m \leq n$.

In this paper, a function $x(n_0, \varphi)$ is said to be a solution of (1) with initial function $\varphi \in S_H$, if $x(n_0, \varphi)$ is a function from $I_r \cup Z^+$ into R^k with the following properties;

(i) $x(n_0, \varphi) \in S_H$ for $n \geq n_0$,

(ii) $x_{n_0+j}(n_0, \varphi) = \varphi(j)$ for $j \in I_r$,

(iii) $x(n_0, \varphi)$ *satisfies* (1) for $n \geq n_0$.

We shall denote by $x(n) \equiv x(n, n_0, \varphi)$ the value of $x(n_0, \varphi)$ at n.

Definition 1. *The zero solution of* (1) *is stable if for any* ε *and any* $n_0 \in Z^+$, *there exists a* $\delta(n_0, \varepsilon) > 0$ *such that if* $\|\varphi\| < \delta$, *then* $|x(n, n_0, \varphi)| < \varepsilon$ *for all* $n \geq n_0$.

Definition 2. *The zero solution of* (1) *is uniformly stable if* δ *of Definition 1 is independent of* n_0.

Definition 3. *The zero solution of* (1) *is uniformly asymptotically stable if it is uniformly stable and there is an* $\eta > 0$ *such that for each* $\gamma > 0$, *there exists an integer* $N(\gamma) > 0$ *independent of* n_0 *such that* *if* $\|\varphi\| < \eta$, *then* $|x(n, n_0, \varphi)| < \varepsilon$ *for all* $n \geq n_0 + N(\gamma)$.

Definition 4. *K denotes the families of continuous increasing, positive definite functions.*

3. Preliminary Results

Theorem 1. *Suppose that there exists a Liapunov functional* $V : Z^+ \times S_H \to R^+$ *satisfying the following conditions ;*

(i) $a(|\varphi(0)|) \leq V(n, \varphi) \leq b(|\varphi(0)|) + c\left(\sum_{j=-1}^{0} d(|\varphi(j)|)\right)$,

where a, b, c *and* $d \in K$,

(ii) $\Delta V_{(1)}(n, x_n(n_0, \varphi)) \leq -e(|x(n)|)$,

where $e \in K$ *and*

$$\Delta V_{(1)}(n, x_n(n_0, \varphi)) = V(n+1, x_{n+1}(n_0, \varphi)) - V(n, x_n(n_0, \varphi)).$$

Then the zero solution of (1) *is uniformly asymptotically stable.*

See, reference [1].

Theorem 2. *Suppose that there exists a Liapunov function* $V : Z \times B_H \to R^+$,
where $B_H = \{|x| < H ; x \in R^k\}$, *such that ;*

(i) $a(|x|) \leq V(n, x) \leq b(|x|)$,

(ii) $\Delta V_{(1)}(n, x(n)) \leq -c(|x(n)|)$ *if* $P[V(n+1, x(n+1))] > V(s, x(s))$ *for* $n - r \leq s \leq n$,

where a, b *and* $c \in K$, *and* $P : R^+ \to R^+$ *is a continuous function with* $P(u) > u$ *if* $u > 0$.
Then the zero solution of (1) *is uniformly asymptotically stable. Moreover, if* $H = \infty$ *and* $a(u) \to \infty$ *as* $u \to \infty$,
then the zero solution of (1) *is a global attractor.*

See, references [1] and [2].

4. Main Results

Theorem 3. *Suppose that there exists a Liapunov functional* $V(n, \varphi)$ *defined on* $Z^+ \times S_H$, *which satisfies the following conditions ;*

(i) $a(n, |\varphi(0)|) \leq V(n, \varphi) \leq b(|\varphi(0)|)$,

where $a(t, u)$ *is continuous in* (t, u), *increases monotonically with respect to* t *and* u, $a(t, u) > 0$ *for* $u \neq 0$, $a(t, 0) \equiv 0$ *and* $b \in K$,

(ii) $\Delta V_{(1)}(n, x_n(n_0, \varphi)) \leq 0$.

Then the zero solution of (1) *is uniformly stable.*

(Proof) For any ε $(0 < \varepsilon < H)$, we can choose a $\delta(\varepsilon)(> 0)$ such that $b(\delta) < a(0, \varepsilon)$. Let any $\varphi \in S_\delta$ and any $n_0 \in Z^+$. Suppose that there exists a $n^*(n_0 < n^*)$ such that $|x(n^*)| \geq \varepsilon$ and $|x(n)| < \varepsilon$ for $n(n_0 < n < n^*)$. By (i) and (ii),

$$\begin{aligned} a(0, \varepsilon) &\leq a(0, |x(n^*)|) \\ &\leq a(n^*, |x(n^*)|) \\ &\leq V(n^*, x_{n^*}(n_0, \varphi)) \\ &\leq V(n_0, x_{n_0}(n_0, \varphi)) \end{aligned}$$

$$\begin{aligned} &\le b(|\varphi(0)|) \\ &< b(\delta) \\ &< a(0, \varepsilon). \end{aligned}$$

This is a contradiction, and hence, if $\varphi \in S_\delta$, then $|x(n)| < \varepsilon$ for any $n \ge n_0$.
This completes the proof of the theorem.

Theorem 4. *Suppose that there exists a Liapunov function $V(n, x)$ defined on $Z \times B_H$, which satisfies the following conditions ;*

(i) $a(n, |x|) \le V(n, x) \le b(|x|)$,

where $a(t, u)$ is continuous in (t, u), increases monotonically with respect to t for each fixed u, $a(t, u) > 0$ for $u \ne 0$, $a(t, 0) \equiv 0$ and $b \in K$,

(ii) $\Delta V_{(1)}(n, x(n)) \le 0$.

Then the zero solution of (1) *is uniformly stable.*

(Proof) For a given $\varepsilon > 0$ $(\varepsilon < H)$, we can choose a $\delta(\varepsilon)\ (> 0)$ such that $b(\delta) < a(0, \varepsilon)$. Let any $\varphi \in S_\delta$ and any $n_0 \in Z^+$. From the condition (ii), there exists a $j^*(n_0 - r \le j^* \le n_0)$ such that

$$V(n, x(n)) \le V(j^*, \varphi(j^*)) \text{ for } n \ge n_0,$$

because,

$$V(n, x(n)) \le V(n_0 + j^*, x(n_0 + j^*)) = V(n_0 + j^*, \varphi(j^*)) \le V(j^*, \varphi(j^*)) \text{ for } -r \le j^* \le 0.$$

By condition (i),

$$\begin{aligned} a(0, |x(n)|) &\le a(n, |x(n)|) \\ &\le V(n, x(n)) \\ &\le V(j^*, \varphi(j^*)) \\ &\le b(|\varphi(j^*)|) \\ &< b(\delta) \\ &< a(0, \varepsilon), \end{aligned}$$

hence, if $\varphi \in S_\delta$, then $|x(n)| < \varepsilon$ for any $n \ge n_0$.
Therefore, the zero solution of (1) is uniformly stable.

References

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