LOCALLY CONFORMAL KAEHLER MANIFOLDS AND REFLECTIONS

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1. Introduction

In [14], I. Vaisman shows that locally conformal symplectic manifolds may be seen as generalized phase spaces of Hamiltonian dynamical systems. Let (M, g, J) be a locally conformal Kaehler manifold i.e., M is a Hermitian manifold and the fundamental 2-form Ω defines a locally conformal symplectic structure on M. There are many studies of submanifods of locally conformal Kaehler manifolds [3], [4], [13]. On the other hand, the consept of the reflection with respect to a submanifold was introduced by B.Y. Chen and L. Vanhecke [1]. In [7] and [8], we studied the reflections with respect to submanifolds and the fibers in a Riemannian submersion.

The almost Hermitian submersion studied by B. Watson [15] and D.L. Johnson [6]. Let (N, g', J') be an almost Hermitian manifold. We shall consider an almost Hermitian submersion $\pi : M \to N$. In this paper, we shall study the reflections with respect to submanifolds of a locally conformal Kaehler manifold M and the reflections with respect to the fibers.

In Section 2 we review basic facts about locally conformal Kaehler manifold, Riemannian submersion and reflection. In Section 3 we gives the main results.

2. Preliminaries

Let (M, g, J) be a Hermitian manifold of complex dimension $n \geq 2$, where g denotes the Hermitian metric and J is the complex structure. Let Ω be its fundamental 2-form, i.e. $\Omega(X, Y) = g(X, JY)$. Then M is a locally conformal Kaehler manifold if:

(1) $\qquad d\Omega = \omega \wedge \Omega,$

for some closed globally defined f-form ω on M [10]. The 1-form ω is the Lee form of M. Next we define a Lee vector field B by

(2) $\qquad g(X, B) = \omega(X).$

Let ∇ be the Levi-Civita connection of g. On M we have another torsionless linear connection ${}^W\nabla$, defined in [10], called the Weyl connection, which is given by

$$(3) \qquad {}^W\nabla_X Y = \nabla_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X, Y)B.$$

It is shown in [10] that an almost Hermitian manifold M is a locally conformal Kaehler if and only if there is a closed 1-form ω on M such that

(4) $\qquad {}^W\nabla_X J = 0.$

The equation (4) is equivalent to

$$(5) \qquad \nabla_X JY - \frac{1}{2}\omega(JY)X + \frac{1}{2}g(X, JY)B = J\nabla_X Y - \frac{1}{2}\omega(Y)JX + \frac{1}{2}g(X, Y)JB,$$

where X and Y are vector fields of M.

Let ${}^WR$, R be respectively the curvature tensor fields of the Weyl connection ${}^W\nabla$, and of ∇. Then

$$
\begin{aligned}
(6) \qquad {}^{W}R(X, Y)Z = R(X, Y)Z - \frac{1}{2}\{&[(\nabla_X\omega)Z + \frac{1}{2}\omega(X)\omega(Z)]Y \\
&- [(\nabla_Y\omega)Z + \frac{1}{2}\omega(Y)\omega(Z)]X - g(Y, Z)(\nabla_X B + \frac{1}{2}\omega(X)B) \\
&+ g(X, Z)(\nabla_Y B + \frac{1}{2}\omega(Y)B)\} - \frac{1}{4}|\omega|^2(g(Y, Z)X - g(X, Z)Y),
\end{aligned}
$$

where X, Y and Z are any vector fields of M [12].

A locally conformal Kaehler manifold (M, J, g) is said to be a generalized Hopf manifold if the Lee form is parallel, i.e., $\nabla\omega = 0$ $(\omega \neq 0)$.

A generalized Hopf manifold is called a P_0K-manifold if its Kaehler metric is flat, i.e., ${}^{W}R(X, Y) = 0$.

In this paper, we consider the case where M is a locally conformal Kaehler manifold which is strongly non-Kaehler in the sense that $d\Omega \neq 0$ (or $\omega \neq 0$) at every point of M.

We recall the following result.

Lemma 1 ([4]). *An invariant submanifold Q of a locally conformal Kaehler manifold M is minimal if and only if the Lee vector field B of M is tangent to Q.*

Proof. By using the normal part of Q in the equation (5), we get the result.

Let M and N be Riemannian manifolds. By a Riemannian submersion we mean a C^∞ mapping $\pi: M \to N$ such that π is of maximal rank and π_* preserves the lengths of horizontal vectors, i.e., vectors orthogonal to the fiber $\pi^{-1}(x)$ for some $x \in N$. Let (M, J, g) and (N, J', g') be almost Hermitian manifolds. A Riemannian submersion $\pi: M \to N$ is called an almost Hermitian submersion if π is an almost complex mapping, i.e., $\pi_* J = J'\pi_*$. An almost complex mapping between complex manifolds is said to be holomorphic. Let $\pi: M \to N$ be an almost Hermitian submersion. If M is a Kaehler manifold, then N is also a Kaehler manifold [15]. In this case, an almost Hermitian submersion is called a Kaehler submersion.

Let M be a Riemannian manifold and Q a submanifold of M. We consider the local diffeomorphism

$\varphi_Q: y \to \varphi_Q(y), \qquad \exp_y(\mathrm{t}X) \mapsto \exp_y(-tX)$

for $X \in T_y{}^{\perp}Q$, $\|X\| = 1$. φ_Q is called the reflection with respect to the submanifold Q. When Q is a point, we obtain the local geodesic symmetry. Let (M, J, g) be an almost Hermitian manifold and Q a submanifold of M. The reflction φ_Q with respect to Q is said to be holomorphic if $\varphi_{Q*} \circ J = J \circ \varphi_{Q*}$.

We assume all data to be analytic. We recall the following results:

Lemma 2 ([1]). *Let M be a Riemannian manifold and Q a submanifold of M. Then the reflection φ_Q is a local isometry if and only if the following conditions satisfied:*

(i) *Q is totally geodesic;*

(ii) $(\nabla^{2k}_{X\ldots X} R)(X, Y, X, V) = 0,$

$(\nabla^{2k+1}_{X\ldots X} R)(X, Y, X, Z) = 0,$

$(\nabla^{2k+1}_{X\ldots X} R)(X, V, X, W) = 0$

for all normal vectors X, Y, Z of Q, all tangent vectors V, W of Q and all $k \in \mathbf{N}$, where R denotes the curvature tensor on M.

Lemma 3 ([1]). *Let (M, g, J) be an almost Hermitian manifold and Q a submanifold such that the* *reflection φ_Q is holomorphic. Then Q is an invariant submanifold and the second fundamental form operator satisfies $\sigma(V, W) = \sigma(JV, JW)$ for all V, W tangent to Q. Moreover, $(\nabla_X J)V$ is normal to Q for all $X \in T^{\perp}Q$, $V \in TQ$.*

3. Results

Let (M, g, J) be a locally conformal Kaehler manifold and Q a submanifold of M. We shall study the reflection φ_Q. In [1], Chen and Vanhecke proved that if M is a Kaehler and the reflection φ_Q is holomorphic, then Q is a totally geodesic submanifold. We shall consider the case where M is a locally conformal Kaehler manifold which is strongly non-Kaehler.

Proposition 1. *Let (M, g, J) be a locally conformal Kaehler manifold and Q a submanifold of M. Let the Lee vector field B be tangent to Q. If the reflection φ_Q is holomorphic, then Q is a totally geodesic submanifold.*

Proof. We recall the Gauss equation $\nabla_V W = \nabla'_V W + \sigma(V, W)$, where V, W are any tangent vector fields of Q. Here ∇' is the induced connection on Q, σ the second fundamental form of Q in M. Since the reflection φ_Q is holomorphic, Q is an invariant submanifold of M. Therefore, since the Lee vector field B is tangent to Q, by (5), we get $\sigma(V, JW) = J\sigma(V, W)$ for any tangent vector fields V, W of Q. Since Q is an invariant submanifold, we have $\sigma(V, W) + \sigma(JV, JW) = 0$. On the other hand, since the reflection φ_Q is holomorphic, by Lemma 3, we get $\sigma(V, W) = \sigma(JV, JW)$. Thus $\sigma = 0$ i.e., Q is a totally geodesic submanifold of M.

Next, we shall consider the case M is a P_0K-manifold.

Proposition 2. *Let M be a P_0K-manifold. Let Q be an invariant submanifold of M. Then the reflection φ_Q is an isometry if and only if Q is a totally geodesic submanifold of M.*

Proof. We assume that Q is a totally geodesic submanifold of M. By Lemma 1, the Lee vector field B is tangent to Q. Since the Lee form is parallel and the curvature ${}^W R = 0$ and $B \in TQ$, we have $R(X, Y, Z, V) = 0$, where X, Y, $Z \in T^{\perp}Q$, $V \in TQ$, and $\nabla R = 0$ [4]. Thus, by Lemma 2, the reflection φ_Q is an isometry. Conversely, if the reflection φ_Q is an isometry, then Q is a totally geodesic submanifold of M.

In the reflection with respect to a submanifold, the following results are known:

(i) When M is a locally symmetric Kaehler manifold and Q is an invariant submanifold of M, the reflection φ_Q is a holomorphic if and only if the reflection φ_Q is an isometry [1].

(ii) Let φ_Q be a holomorphic or anti-holomorphic reflection with respect to a submanifold Q of a quasi-Kaehler manifold M. Then the reflection φ_Q is an isometry [2].

We shall consider the case where M is a locally conformal Kaehler which is strongly non-Kaehler. From the above two Propositions, we have the following result.

Proposition 3. *Let M be a P_0K-manifold and Q a submanifold of M and the Lee vector field $B \in TQ$. If the reflection φ_Q is holomorphic, then the reflection φ_Q is an isometry.*

Finally, we shall study an almost Hermitian submersion. Let $\pi : M \to N$ be an almost Hermitian submersion. It is known that the horizontal and vertical distributions are J-invariant, because π is almost complex [15]. Let M be a P_0K-manifold and N an almost Hermitian manifold. As an application of Proposition 2, we have

Proposition 4. *Let M be a P_0K-manifold and N an almost Hermitian manifold. Let $\pi : M \to N$ be an almost Hermitian submersion. Then the reflections $\varphi_{\pi^{-1}(x)}$ are isometries if and only if $\pi^{-1}(x)$ are totally geodesic submanifolds of M.*

Let M and N be Kaehler manifolds. It is known that if $\phi : M \to N$ is a holomorphic mapping, then $\phi : M \to N$ is a harmonic mapping. In a Riemannian submersion $\pi : M \to N$, π is a harmonic mapping if and only if the fibers are minimal submanifolds. Now, let M be a locally conformal Kaehler manifold and N an almost Hermitian manifold. We consider an almost Hermitian submersion $\pi : M \to N$. Using Lemma 1, we have

Proposition 5. *Let M be a locally conformal Kaehler manifold and N an almost Hermitian manifold. Let $\pi : M \to N$ be an almost Hermitian submersion. Then π is a harmonic mapping if and only if the Lee vector field B is tangent to the fibers.*

4. Examples

We give examples of almost Hermitian submersion and isometric reflection.

Example 1 ([5]). Let H^n_λ be the Hopf manifold and $P_{n-1}(\mathbf{C})$ complex projective space. Let $q : \mathbf{C}^n - \{0\} \to P_{n-1}(\mathbf{C})$ be the natural map. Then q is an almost complex mapping. Let $p : \mathbf{C}^n - \{0\} \to H^n_\lambda = (\mathbf{C}^n - \{0\})/\Delta_\lambda$ be the natural surjection. Since Δ_λ is properly discontinuous group, p is an almost complex mapping. Define $\pi : H^n_\lambda \to P_{n-1}(\mathbf{C})$ by $\pi(p(z)) = [z]$, where $[z]$ denotes the point of $P_{n-1}(\mathbf{C})$ of homogeneous coordinates z. Then $q = \pi \circ p$ and π is an almost Hermitian submersion. The fiber is the complex 1-torus $T^1_{\mathbf{C}}$. Moreover, its fibers are minimal and the Lee vector field of H^n_λ is tangent to the fibers.

Example 2. Let M be a P_0K-manifold. Let ω and B be the Lee form and the Lee vector field respectively. We set $A = JB$ and $\theta = -\omega \circ J$. Let ε be a complex analytic 1-dimensional foliation generated by A, B. The metric of M can be expressed as

(7) $$ds^2 = 2g_{a\bar{b}}dz^a \otimes d\bar{z}^b + (\omega + \sqrt{-1}\theta) \otimes (\omega - \sqrt{-1}\theta).$$

Then the space $N = M/\varepsilon$ of the leaves of ε is a Kaehler manifold, with the metric induced by the first term of (7). Therefore, we have an almost Hermitian submersion $\pi : M \to N$. Since $\nabla_A B = \nabla_B A = \nabla_A A = \nabla_B B = 0$, the fibers are totally geodesic [13]. On the other hand, since M is a P_0 K-manifold and B is vertical, we have $R(X, Y, Z, V) = 0$, where X, Y, Z are horizontal and V is vertical, and $\nabla R = 0$. Hence, by Lemma 2, the reflections $\varphi_{\pi^{-1}(x)}$ with respect to the fibers are isometries.

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