On the Partially Uniform Lipschitz Stability of Systems of Ordinary Differential Equations

Masamichi ASO, Shoichi SEINO and Miki KUDO

(Received on 22 August, 1994)

1. Introduction

In 1989, F.M. Dannan and S. Elaydi introduced a new notion of stability, which will be called uniform Lipschitz stability, for systems of differential equations. This new notion lies somewhere between uniform stability on one side and the notions of asymptotic stability in variation of Brauer [4] and uniform stability in variation of Brauer and Strauss [3] on the other side. The notion of uniform Lipschitz stability is new only as nonlinear systems, since it is equivalent to uniform stability in linear systems [1].

An important feature of uniform Lipschitz stability is that, unlike uniform stability, the linearlized systems inherit the property of uniform Lipschitz stability from the original nonlinear systems.

In [2], F.M. Dannan and S. Elaydi studied the uniform Lipschitz stability theorem using the techniques of Liapunov's second method.

In 1991, M. Kudo showed some generalization of this theorem [5].

In many applications, we need to see the qualities not of the whole solutions but of partial.

The main purpose of this paper is to introduce a new notion of uniformly Lipschitz stability, which we named partially uniform Lipschitz stability, for systems of differential equations, and to state the partially uniform Lipschitz stability theorem using the techniques of Liapunov function satisfying a weak condition.

2. Definitions and Notations

Let I and R^+ denote the intervals $[t_0, \infty)$ and $[0, \infty)$ respectively. And let R^n denote Euclidean n-space.

For $x \in R^n$, let the norm of x be $\|x\|$. We shall denote by $C(I\times R^n\times R^m, R^k)$ the set of all continuous function f defined on $I\times R^n\times R^m$ with value in R^k.

Let $F(t, x) \in C(I\times D, R^n)$ and $F(t, 0) = 0$, where D is an open set in R^n. For systems

$$\frac{dx}{dt} = F(t, x), \quad \cdots\cdots(1)$$

a solution through a point $(t_0, x_0) \in I\times D$ will be denoted by such a form as $x(t, t_0, x_0)$.

Let $f(t, x, y) \in C(I\times R^n\times R^m, R^n)$, $f(t, 0, 0) = 0$ and $g(t, x, y) \in C(I\times R^n\times R^m, R^m)$, $g(t, 0, 0) = 0$.

We consider systems of differential equations

$$\begin{cases} \dfrac{dx}{dt} = f(t, x, y) \\ \dfrac{dy}{dt} = g(t, x, y). \end{cases} \quad \cdots\cdots(2)$$

Further, we consider a scalar differential equation

$$\frac{du}{dt} = \phi(t, u), \cdots\cdots (3)$$

where $\phi(t, u) \in C(I \times R^+, R)$, $\phi(t, 0) = 0$ and $u(t, t_0, u_0) = u(t)$ is a maximal solution of (3) with $u(t_0, t_0, u_0) = u_0$.

We introduce the following definitions.

[**Definition 1**] The zero solution of (1) is said to be uniformly Lipschitz stable if there exist $M \geqq 1$ and $\delta > 0$ such that $\|x(t, t_0, x_0)\| \leqq M\|x_0\|$ for $\|x_0\| < \delta$ and $\forall t \geqq t_0 \geqq 0$.

[**Definition 2**] The zero solution of (3) is said to be uniformly Lipschitz stable if there exist $M \geqq 1$ and $\delta > 0$ such that $u(t, t_0, u_0) \leqq Mu_0$ for $u_0 < \delta$ and $\forall t \geqq t_0 \geqq 0$.

[**Definition 3**] The zore solution of (2) is said to be partially uniformly Lipschitz stable with respect to x if there exist $M \geqq 1$ and $\delta > 0$ such that $\|x(t, t_0, x_0, y_0)\| \leqq M(\|x_0\| + \|y_0\|)$ for $\|x_0\| + \|y_0\| < \delta$ and $\forall t \geqq t_0 \geqq 0$.

[**Definition 4**] Corresponding to $V(t, x) \in C(I \times R^n, R^+)$ and $V(t, x, y) \in C(I \times R^n \times R^m, R^+)$, we define the functions

$$V'_{(1)}(t, x) = \lim_{h \to 0} \sup \frac{1}{h} \{V(t+h, x+hF(t, x)) - V(t, x)\}$$

and

$$V'_{(2)}(t, x, y) = \lim_{h \to 0} \sup \frac{1}{h} \{V(t+h, x+hf(t, x, y), y+hg(t, x, y)) - V(t, x, y)\}$$

respectively. If $V(t, x)$ and $V(t, x, y)$ satisfy locally Lipschitz condition with respect to x and (x, y), then $V'_{(1)}(t, x) = V'(t, x)$ and $V'_{(2)}(t, x, y) = V'(t, x, y)$ respectively.

In case $V(t, x)$ and $V(t, x, y)$ have continuous partial derivatives of the first order, it is evident that

$$V'_{(1)}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot F(t, x)$$

and

$$V'_{(2)}(t, x, y) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t, x, y) + \frac{\partial V}{\partial y} \cdot g(t, x, y),$$

where "$\cdot$" denotes a scalar product.

3. Preliminary results

[Theorem 1] Suppose that the maximal solution $u(t)$ of (3) such that $u(t_0) = u_0$ stays on interval $[a, b]$.

If a continuous function $V(t)$ with $V(t_0) \leqq u_0$ satisfies

$$V'(t) \leqq \phi(t, V(t)),$$

where $\phi(t, u)$ is continuous on an open connected set $\Omega \in R^2$, then we have

$$V(t) \leqq u(t) \text{ for } a \leqq t \leqq b.$$

For the proof of this theorem, see references [6], [7], [8].

[Theorem 2] Suppose that there exist two functions $V(t, x)$ and $\phi(t, u)$ satisfying the following conditions;

(i) $\phi \in C(I \times R^+, R)$ and $\phi(t, 0) = 0$,

(ii) $V(t, x) \in C(I \times S, R^+)$, where $S = \{x \mid \|x\| < \rho, x \in R^n\}$, $V(t, 0) = 0$, $V(t, x)$ is locally Lipschitz in x and satisfies

$$b(\|x\|) \leqq V(t, x),$$

where $b(r) \in C([0, \rho], R^+)$, $b(0) = 0$, and $b(r)$ is strictly monotone increasing in r such that $b^{-1}(\alpha r) \leqq rq(\alpha)$ for some function q, with $q(\alpha) \geqq 1$ if $\alpha \geqq 1$,

(iii) $V'_{(1)}(t, x) \leqq \phi(t, V(t, x))$, where $(t, x) \in I \times S$.

If the zero solution of (1) is uniformly Lipschitz stable, then so is the zero solution of (1).

For the proof of this theorem, see reference [2].

[Theorem 3] Suppose that there exist functions $V \in C(I \times R^n, R^+)$, $a \in C(I \times R^+, R^+)$, $c \in C(I \times R^+, R^+)$ and $\phi \in C(I \times R^+, R)$ such that

(i) $V(t, x)$ is locally Lipschitz in x and $V(t, 0) = 0$,

(ii) $a(t, \|x\|) \leqq V(t, x) \leqq c(t, \|x\|)$,

where $a(t, r)$ increases monotonically with respect to t for each fixed r, $a(t, 0) = 0$, $a(t, r) > 0$ for $r \neq 0$, $kc(t, s) \leqq c(t, ks)$ for a positive constant k and if $a(t, r) \leqq c(t, s)$, then $r \leqq s$,

(iii) $V'_{(1)}(t, x) \leqq \phi(t, V(t, x))$.

If the zero solution of (3) is uniformly Lipschitz stable, then the zero solution of (1) is uniformly Lipschitz stable.

For the proof of this theorem, see reference [5].

4. Main result

[Theorem 4] Suppose that there exist functions $V \in C(I \times R^n \times R^m, R^+)$, $a \in C(I \times R^+, R^+)$, $b \in C(I \times R^+, R^+)$ and $\phi \in C(I \times R^+, R)$ such that

(i) $V(t, 0, 0) = 0$,

(ii) $a(t, \|x\|) \leqq V(t, x, y) \leqq b(t, \|x\| + \|y\|)$,

where $a(t, r)$ increases monotonically with respect to t for each fixed r, $a(t, 0) = 0$, $a(t, r) > 0$ for $r \neq 0$, $kb(t, s) \leqq b(t, ks)$ for a positive constant k and if $a(t, r) \leqq b(t, s)$, then $r \leqq s$,

(iii) $V'_{(2)}(t, x, y) \leqq \phi(t, V(t, x, y))$.

If the zero solution of (3) is uniformly Lipschitz stable, then the zero solution of (2) is partially uniformly Lipschitz stable with respect to x.

[**Proof**] From the uniform Lipschitz stability of the zero solution $u = 0$ of (3), there exist $\delta > 0$ and some constant $M \geqq 1$ such that $u(t, t_0, u_0) \leqq Mu_0$, whenever $u_0 < \delta$. For δ, there exists $\delta_1 > 0$ such that if $\|x_0\| + \|y_0\| \leqq \delta_1$, $V(t_0, x_0, y_0) \leqq \delta$ by using $V \in C$ and (i). Therefore if we put $V(t_0, x_0, y_0) = u_0$, we have $u_0 < \delta$ and $u(t, t_0, u_0) \leqq Mu_0$. Using the comparison theorem, from the condition (iii), we have

$$V(t, x(t), y(t)) \leqq u(t, t_0, u_0).$$

Hence, by the condition (ii), we have

$$\begin{aligned} a(t_0, \|x(t, t_0, x_0, y_0)\|) &\leqq a(t, \|x(t, t_0, x_0, y_0)\|) \\ &\leqq V(t, x(t), y(t)) \\ &\leqq u(t, t_0, u_0) \\ &\leqq Mu_0 \\ &= MV(t_0, x_0, y_0) \\ &\leqq Mb(t_0, \|x_0\| + \|y_0\|) \\ &\leqq b(t_0, M(\|x_0\| + \|y_0\|)). \end{aligned}$$

Thus we have $\|x(t, t_0, x_0, y_0)\| \leqq M(\|x_0\| + \|y_0\|)$ which shows that the zero solution of (2) is partially uniformly Lipschitz stable.

References

[1] F.M. Dannan and S. Elaydi : Lipschitz Stability of Nonlinear Systems of Differential Equations, J. Math. Anal. Appl., 113 (1986), 562-577.
[2] F.M. Dannan and S. Elaydi : Lipschitz Stability of Nonlinear Systems of Differential Equations II, Liapunov Functions, J. Math. Anal. Appl., 143 (1989), 517-529.
[3] F. Brauer and A. Strauss : Perturbations of Nonlinear Systems of Differential Equations III, J. Math. Anal. Appl., 31 (1970), 37-48.
[4] F. Brauer : Perturbations of Nonlinear Systems of Differential Equations IV, J. Math. Anal. Appl., 37 (1972), 214-222.
[5] M. Kudo : On the Uniformly Lipschitz Stability of Nonlinear Differential Equations by the Comparison Principle, Research Reports of Akita National College of Technology, 26 (1991), 41-44.
[6] V. Lakshmikantham and S. Leela : Differential and Integral Inequalities, Academic Press, 1969.
[7] A. Halanay : Differential Equations ; Stability, Oscillations, Time Lag, Academic Press, 1966.
[8] T. Yoshizawa : Stability Theory by Liapunov's second method, Math. Soc. Japan, 1966.
[9] T. Yoshizawa : Stability Theory and the Existance of Periodic Solutions and Almost Periodic Solutions, Springer-Verlag, 1975.

平成6年11月