An Optimal Control by The Pontryagin Maximum Principle

—— Terminal Control Problem ——

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Let us consider the system of differential equations

$$\left.\begin{aligned} \dot{x}^1(t) &= x^2(t), \\ \dot{x}^2(t) &= -x^1(t) - x^2(t) + u(t), \end{aligned}\right\} \qquad (1)$$

under the hypothesis that the initial condition is

$$x^i(0) = x_o^i \qquad (i=1,2). \qquad (2)$$

In addition, let the control region U be defined by the inequality

$$|u(t)| \leqq 1. \qquad (3)$$

We want to determine the optimal control $u^*(t)$ which minimizes the deviation

$$e(T) = |x^1(T)|, \qquad (4)$$

at the terminal point of the trajectory, where T is fixed.

We shall consider the application of the Pontryagin maximum principle to the solution of the above problem.

Now we introduce a new variable

$$x^0(t) = e(t) = |x^1(t)|. \qquad (5)$$

The problem which minimizes $e(T)$ is equivalent to minimizing $x^0(T) = |x^1(T)|$ under conditions (1) and (3).

The system of differential equations with the additional variable is the form

$$\left.\begin{aligned} \dot{x}^0(t) &= \begin{cases} \dot{x}^1(t) = x^2(t), & \text{if } x^1(t) > 0, \\ -\dot{x}^1(t) = -x^2(t), & \text{if } x^1(t) < 0, \end{cases} \\ \dot{x}^1(t) &= x^2(t), \\ \dot{x}^2(t) &= -x^1(t) - x^2(t) + u(t), \end{aligned}\right\} \qquad (6)$$

where the initial conditions are:

$$x^0(0) = e_0, \quad x^i(0) = x_o^i, \quad (i=1,2). \qquad (7)$$

The Hamiltonian function H in this case has the form

$$H(p, x, u) = -p_2 x^1 + (\pm p_0 + p_1 - p_2) x^2 + p_2 u. \qquad (8)$$

From (8), the condition which maximizes H with respect to u is

$$p_2 u > 0. \qquad (9)$$

Taking (9) and the condition (3) into account, we have

$$u(t)=\begin{cases} 1\ , & \text{if } p_2>0\ , \\ -1\ , & \text{if } p_2<0\ . \end{cases} \tag{10}$$

Furthermore, we obtain the system of equations

$$\left.\begin{aligned} \dot{p}_0 &= -\frac{\partial H}{\partial x^0} = 0\ , \\ \dot{p}_1 &= -\frac{\partial H}{\partial x^1} = p_2\ , \\ \dot{p}_2 &= -\frac{\partial H}{\partial x^2} = \mp p_0 - p_1 + p_2\ , \end{aligned}\right\} \tag{11}$$

for the auxiliary variables p_0, p_1 and p_2, where the boundary condition is

$$p_i(T)=0, \qquad (i=1,2)\ . \tag{12}$$

We solve (11) : $\quad p_0(t)=\lambda, \tag{13}$

where λ is a non positive constant, and

$$(p_1\pm\lambda)^{\cdot\cdot}-(p_1\pm\lambda)^{\cdot}+(p_1\pm\lambda)=0\ . \tag{14}$$

The general solution of this equation can be written in the form

$$\left.\begin{aligned} p_1(t) &= e^{\frac{1}{2}t}\left(c_1\cos\frac{\sqrt{3}}{2}t + c_2\sin\frac{\sqrt{3}}{2}t\right)\mp\lambda\ , \\ p_2(t) &= \frac{1}{2}e^{\frac{1}{2}t}\left(c_1\left(\cos\frac{\sqrt{3}}{2}t-\sqrt{3}\sin\frac{\sqrt{3}}{2}t\right)+c_2\left(\sqrt{3}\cos\frac{\sqrt{3}}{2}t+\sin\frac{\sqrt{3}}{2}t\right)\right). \end{aligned}\right\} \tag{15}$$

By virtue of (12), we determine c_1 and c_2 :

$$\left.\begin{aligned} c_1 &= \pm\frac{\lambda}{\sqrt{3}}e^{-\frac{1}{2}T}\left(\sqrt{3}\cos\frac{\sqrt{3}}{2}T+\sin\frac{\sqrt{3}}{2}T\right)\ , \\ c_2 &= \mp\frac{\lambda}{\sqrt{3}}e^{-\frac{1}{2}T}\left(\cos\frac{\sqrt{3}}{2}T-\sqrt{3}\sin\frac{\sqrt{3}}{2}T\right)\ . \end{aligned}\right\} \tag{16}$$

Consequently

$$p_2(t)=\mp\frac{4\lambda}{\sqrt{3}}e^{\frac{1}{2}(t-T)}\sin\left(\frac{\sqrt{3}}{2}(t-T)\right)\ . \tag{17}$$

Taking (10) and $\lambda\leqq 0$ into account, we have two cases as follows.

Case (I). When $x^1(t)>0$,

$$u^*(t)=1\ , \quad \text{if } \sin\left(\frac{\sqrt{3}}{2}(t-T)\right)>0\ ,$$

$$u^*(t)=-1\ , \quad \text{if } \sin\left(\frac{\sqrt{3}}{2}(t-T)\right)<0\ ,$$

i.e.,

$$\left.\begin{aligned} u^*(t) &= 1\ , && \text{if } \frac{4}{\sqrt{3}}m\pi+T<t<\frac{2}{\sqrt{3}}(2m+1)\pi+T\ , \\ u^*(t) &= -1\ , && \text{if } \frac{2}{\sqrt{3}}(2m-1)\pi+T<t<\frac{4}{\sqrt{3}}m\pi+T\ . \end{aligned}\right\} \tag{18}$$

Case(II). When $x^1(t)<0$,

$$u^*(t)=1\ , \quad \text{if } \sin\left(\frac{\sqrt{3}}{2}(t-T)\right)<0\ ,$$

$$u^*(t)=-1\ , \quad \text{if } \sin\left(\frac{\sqrt{3}}{2}(t-T)\right)>0\ ,$$

i.e.,
$$u^*(t)=\ \ 1\ ,\quad \text{if}\ \ \frac{2}{\sqrt{3}}(2m-1)\pi+T<t<\frac{4}{\sqrt{3}}m\pi+T$$
$$u^*(t)=-1\ ,\quad \text{if}\ \ \frac{4}{\sqrt{3}}m\pi+T<t<\frac{2}{\sqrt{3}}(2m+1)\pi+T. \tag{19}$$

As we have seen above, it is obvious that the switching occurs every $\frac{2\pi}{\sqrt{3}}$ units of time.

Let us solve the system of equations (6) and study the segments of trajectory corresponding to a certain time interval on which $u^*(t)=1$ and $u^*(t)=-1$.

Next we shall consider the synthesizing the optimal trajectory of the system (6).

From (6), we obtain
$$(x^1\mp 1)^{\cdot\cdot}+(x^1\mp 1)^{\cdot}+(x^1\mp 1)=0\ , \tag{20}$$
corresponding to $u^*(t)=\pm 1$.

We can immediately find the general solution of this system :
$$x^1(t)=e^{-\frac{1}{2}t}(c_1\cos\frac{\sqrt{3}}{2}t+c_2\,\text{siu}\,\frac{\sqrt{3}}{2}t)\pm 1\ ,$$
$$x^2(t)=-e^{-\frac{1}{2}t}(c_1\sin(\frac{\sqrt{3}}{2}t+\frac{\pi}{6})-c_2\cos(\frac{\sqrt{3}}{2}t+\frac{\pi}{6}))\ . \tag{21}$$

According to the initial condition $x^i(0)=x_0^i$, $(i=1,2)$, we decide c_1 and c_2
$$c_1=x_0^1\mp 1\ ,\quad c_2=\frac{1}{\sqrt{3}}(2x_0^2+x_0^1\mp 1)\ . \tag{22}$$

Since it is difficult to get the synthesis by means of eliminating the parameter t, we shall discuss the solution which is transformed into the polar coodinate.

First of all we consider the homogeneous system
$$\dot{x}^1(t)=\ x^2(t)\ ,$$
$$\dot{x}^2(t)=-x^1(t)-x^2(t)\ . \tag{23}$$

By means of a linear transformation
$$y^1=-x^1-\frac{1}{2}x^2\ ,$$
$$y^2=\frac{\sqrt{3}}{2}x^2\ , \tag{24}$$

system (23) can be reduced to the form
$$\dot{x}^1(t)=-\frac{1}{2}x^1(t)-\frac{\sqrt{3}}{2}x^2(t)\ ,$$
$$\dot{x}^2(t)=\frac{\sqrt{3}}{2}x^1(t)-\frac{1}{2}x^2(t)\ . \tag{25}$$

Its general solution is :

$$\left.\begin{aligned} x^1(t) &= ce^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t+\alpha\right) , \\ x^2(t) &= ce^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t+\alpha\right) , \end{aligned}\right\} \tag{26}$$

where c and α are constants of integration.

we put
$$\begin{aligned} x^1 &= \rho \cos \varphi , \\ x^2 &= \rho \sin \varphi , \end{aligned} \tag{27}$$

then (26) is :
$$\begin{aligned} \rho &= ce^{-\frac{1}{2}t} , \\ \varphi &= \frac{\sqrt{3}}{2}t+\alpha . \end{aligned} \tag{28}$$

Eliminating the parameter t, we have the polar equation of the phase trajectory

$$\rho = Ke^{-\frac{1}{\sqrt{3}}\varphi} , \tag{29}$$

where $K = ce^{\frac{1}{\sqrt{3}}\alpha}$.

Next we consider the non homogeneous system

$$\left.\begin{aligned} \dot{x}^1(t) &= -\frac{1}{2}x^1(t) - \frac{\sqrt{3}}{2}x^2(t) + v^1 , \\ \dot{x}^2(t) &= \frac{\sqrt{3}}{2}x^1(t) - \frac{1}{2}x^2(t) + v^2 , \end{aligned}\right\} \tag{30}$$

where
$$\left.\begin{aligned} v^1 &= 0 , \\ v^2 &= u . \end{aligned}\right\} \tag{31}$$

It follows from (31) that the point $v=(v^1, v^2)$ describes a segment V with the endpoints e_1 and e_2 whose coordinates are :

$$e_1=(0,-1) , \quad e_2=(0,1) . \tag{32}$$

We shall denote by $w=g(v)$ the point whose coordinate (w^1, w^2) satisfies the relations

$$\left.\begin{aligned} -\frac{1}{2}w^1 - \frac{\sqrt{3}}{2}w^2 + v^1 &= 0 , \\ \frac{\sqrt{3}}{2}w^1 - \frac{1}{2}w^2 + v^2 &= 0 , \end{aligned}\right\} \tag{33}$$

for any point $v=(v^1, v^2)$ in the phase plane.

The affine transformation g translates from the segment V to the segment W with the endpoints h_1 and h_2.

From the relations

$$\begin{aligned} -\frac{1}{2}h_i^1 - \frac{\sqrt{3}}{2}h_i^2 + e_i^1 &= 0 , \\ \frac{\sqrt{3}}{2}h_i^1 - \frac{1}{2}h_i^2 + e_i^2 &= 0 , \end{aligned} \tag{34}$$

the coordinate (h_i^1, h_i^2) of the endpoint h_i $(i=1,2)$ is determined.

We get

$$h_1 = (\frac{\sqrt{3}}{2}, -\frac{1}{2}), \qquad h_2 = (-\frac{\sqrt{3}}{2}, \frac{1}{2}). \tag{35}$$

If the optimal control v takes on the value e_i $(i=1,2)$ on a certain time interval, then the system of equations of the object is the form

$$\left.\begin{aligned} \dot{x}^1(t) &= -\frac{1}{2}x^1(t) - \frac{\sqrt{3}}{2}x^2(t) + e_i^1, \\ \dot{x}^2(t) &= \frac{\sqrt{3}}{2}x^1(t) - \frac{1}{2}x^2(t) + e_i^2 \end{aligned}\right\} \quad (i=1,2) \tag{36}$$

It follows from (34) and (36) that the system (30) is described by the equations

$$\left.\begin{aligned} (x^1(t)-h_i^1)^{\cdot} &= -\tfrac{1}{2}(x^1(t)-h_i^1) - \frac{\sqrt{3}}{2}(x^2(t)-h_i^2), \\ (x^2(t)-h_i^2)^{\cdot} &= \frac{\sqrt{3}}{2}(x^1(t)-h_i^1) - \tfrac{1}{2}(x^2(t)-h_i^2) \end{aligned}\right\} \tag{37$_i$}$$

We draw two half-lines which is perpendicular to the segment V passing through the origin. Let $\alpha_i(=\pi)$ be the angle formed by these lines. We denote by p_i any non zero vector which lies in the region defined by the angle.

In this case the Hamiltonian H to the system (30) is the form

$$H = (-\frac{1}{2}p_1 + \frac{\sqrt{3}}{2}p_2)x^1 + (-\frac{\sqrt{3}}{2}p_1 - \frac{1}{2}p_2)x^2 + p_1v^1 + p_2v^2, \tag{38}$$

so that it is obvious that H attains its maximum simultaneously with $p_1v^1+p_2v^2$.

The system of equations for the auxiliary variable p_i is the following form :

$$\left.\begin{aligned} \dot{p}_1 &= -\frac{\partial H}{\partial x^1} = \frac{1}{2}p_1 - \frac{\sqrt{3}}{2}p_2, \\ \dot{p}_2 &= -\frac{\partial H}{\partial x_2} = \frac{\sqrt{3}}{2}p_1 + \frac{1}{2}p_2. \end{aligned}\right\} \tag{39}$$

Its general solution is :

$$\left.\begin{aligned} p_1(t) &= ke^{\frac{1}{2}t}\cos(\frac{\sqrt{3}}{2}t+\beta), \\ p_2(t) &= ke^{\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t+\beta), \end{aligned}\right\} \tag{40}$$

where k and β are constants of integration.

It follows immediately from (40) that the switching occurs every $\frac{2\pi}{\sqrt{3}}$.

Now we can construct the switching curves in the phase plane which determine the synthesis of the optimal control.

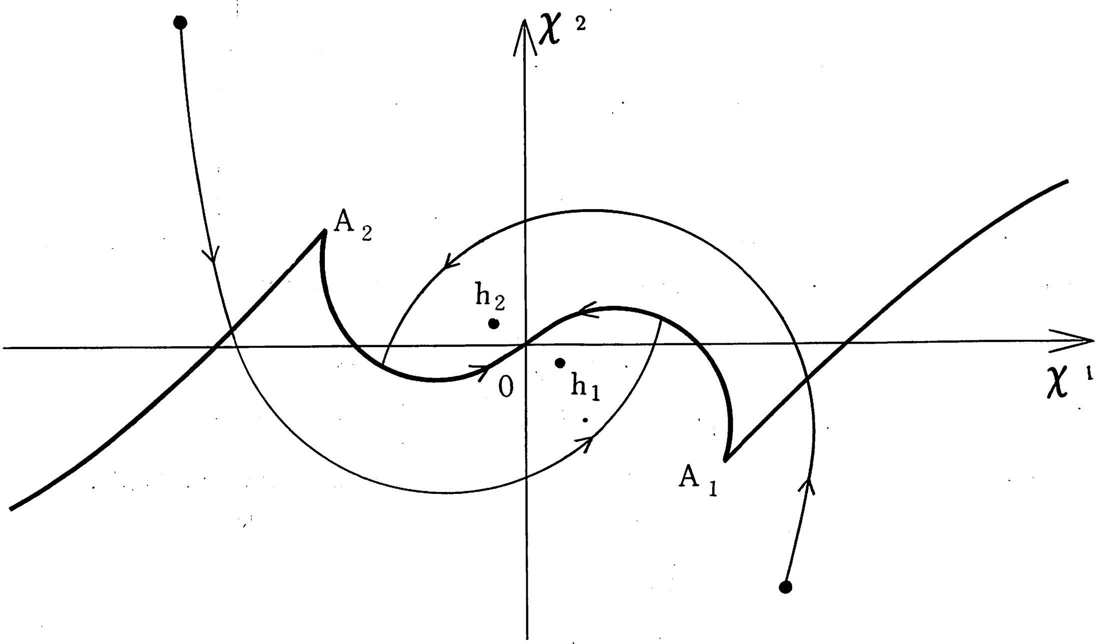
Let P_i be the similarity transformation with center at h_i and ratio $e^{\frac{\alpha}{\sqrt{3}}}$, accompanied by a clockwise rotation about h_i through the angle α_i.

Let A_i O be the arc of the trajectory of system $(37)_i$ which terminates at the point O, and which is spaned toward the equilibrium point by the angle α_i.

Then the arc $B_{i-1}A_{i-1}$ can be obtained from the arc A_iO with the aid of a transformation P_{i-1}.

Proceeding in this way, let us draw the two curves $OA_iB_iC_i\cdots\cdots$, $(i=1, 2)$, piecewise smoothing curves, which start at the origin, and which represent the locus of the switching points.

Thus, the synthesis of the optimal control and the form of the optimal trajectory are showen in the next figure.



Everything that was said above refers to the synthesis of the optimal control in the plane transformed by (24), so that the picture of the synthesis of the optimal control of the original x^1, x^2 plane is obtained by this affine transformation.

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〔References〕

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