# Solution of a Minimum Effort Control Problem

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The maximum principle introduced by L. S. Pontryagin et al. offers us an effective means for solving optimal control problems, but it is not an algorithm for getting optimal controls and their syntheses. Therefore, solving a given optimal control problem, we must take some suitable devices for it using the maximum principle.

We shall take the so-called minimum effort control problem and apply the maximum principle.

PROBLEM:

Suppose that the object's law of motion is written by a differential equation of the second order

$$\ddot{x}(t)=u(t), \tag{1}$$

and a control u(t) is satisfied the constraint $|u(t)|\leq M$.

Seek to find an optimal control u(t) which transfers the object from x(0) to the origin so as to minimize an effort

$$J=\int_0^T |u(t)|\,dt, \tag{2}$$

where T is not fixed.

SOLUTION:

We don't deal with the equation (1) immediately, but the system of two differential equations of the first order:

$$\dot{x}^1(t)=x^2(t), \quad \dot{x}^2(t)=u(t). \tag{3}$$

In order to solve the problem at hand, it is convenient to consider the transition of the phase point $(x^1(t), x^2(t))$ on the phase space (in this case, it is two dimensional Euclidean space). we put the initial conditions

$$x^1(0)=x^1_0, \quad x^2(0)=x^2_0, \tag{4}$$

and the terminal conditions

$$x^1(T)=0, \quad x^2(T)=0. \tag{5}$$

We introduce an additional function $x^0(t)$ of t and translate the equation (2) into

$$\dot{x}^0(t)=|u(t)|, \quad x^0(0)=x^0_0=0. \tag{6}$$

The problem under consideration is therefore equivalent to that of selecting an admissible control u(t) so as to minimize $x^0(T)$ satisfying the system (3) and the boundary conditions (4) and (5), and consequently we can apply the maximum principle.

Using auxiliary variables $p_0(t)$, $p_1(t)$ and $p_2(t)$, the Hamiltonian function H is written in the form

$$H(p(t), x(t), u) = p_0\dot{x}^0 + p_1\dot{x}^1 + p_2\dot{x}^2 = p_0|u| + p_1x^2 + p_2u. \tag{7}$$

According to the Pontryagin maximum principle, if our control process $(u(t), x(t))$ is optimal, it is necessary that there exist a constant $p_0 = \lambda \leqq 0$ and a non-trivial solution of the system

$$\dot{p}_i(t) = -\frac{\partial H}{\partial x^i}\ ; \quad i=1,2, \tag{8}$$

being satisfied the maximum value condition

$$\max_u H(p(t), x(t), u) = H(p(t), x(t), u(t)) = 0 \tag{9}$$

with respect to u at all regular points of $u(t)$. Hence

$$\dot{p}_0 = 0, \quad \dot{p}_1 = 0, \quad \dot{p}_2 = -p_1. \tag{10}$$

From the maximum value condition, control $u(t)$ must be satisfied

$$u(t) = \begin{cases} -M & \text{if } p_2 < \lambda, \\ 0 & \text{if } \lambda < p_2 < -\lambda, \\ M & \text{if } p_2 > -\lambda, \end{cases} \tag{11}$$

because of the constraint $|u(t)| \leqq M$ (Rf. Fig. 1). Moreover, from (10) we obtain

$$p_1(t) = a, \quad p_2(t) = -at + b, \tag{12}$$

where a and b are constants.

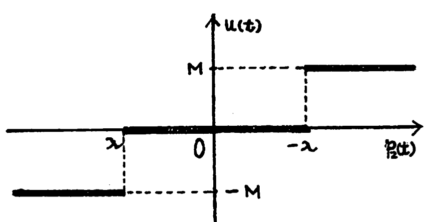

Fig. 1

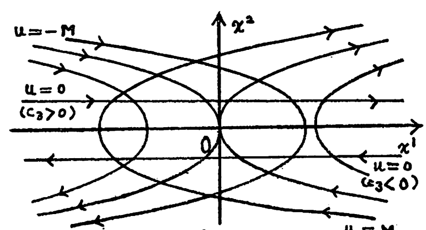

Fig. 2

From the system (3) we obtain using integration constants,

if $u=M$, $$x^1(t) = \frac{1}{2}Mt^2 + c_1t + c_2, \quad x^2(t) = Mt + c_1, \tag{13}$$

and the corresponding trajectory is

$$x^1 = \frac{1}{2M}\left\{(x^2)^2 - c_1^2\right\} + c_2, \tag{14}$$

if $u=0$, $$x^1(t) = c_3t + c_4, \quad x^2(t) = c_3, \tag{15}$$

i.e. $$x^2 = c_3, \tag{16}$$

if $u=-M$, $$x^1(t) = -\frac{1}{2}Mt^2 + c_5t + c_6, \quad x^2(t) = -Mt + c_5, \tag{17}$$

i.e. $$x^1 = -\frac{1}{2M}\left\{(x^2)^2 - c_5^2\right\} + c_6. \tag{18}$$

The trajectories of (14), (16) and (18) are shown in Fig.2.

Judging from the direction of motion of the phase point, the switching curve is $x^1 = -(x^2)^2/2M$, if $x^2>0$, and $x^1 = (x^2)^2/2M$, if $x^2<0$. By the curve and a straight line $x^2=0$, the phase space is divided into four regions as follows (Rf. Fig. 3):

$$\text{I} : x^1 \geqq \frac{1}{2M}(x^2)^2 \ ; \ x^2<0, \quad (19)$$

$$\text{II} : x^1 > -\frac{1}{2M}(x^2)^2 \ ; \ x^2 \geqq 0, \quad (20)$$

$$\text{III} : x^1 \leqq -\frac{1}{2M}(x^2)^2 \ ; \ x^2>0, \quad (21)$$

$$\text{IV} : x^1 < \frac{1}{2M}(x^2)^2 \ ; \ x^2 \leqq 0, \quad (22)$$

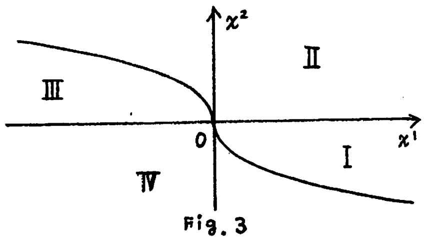

Fig. 3

From the equation (11), we obtain that the optimal control u(t) switches its value as follows:

1) if $(x^1_0, x^2_0) \in \text{I}$, $0 \longrightarrow M$,
2) if $(x^1_0, x^2_0) \in \text{II}$, $-M \longrightarrow 0 \longrightarrow M$,
3) if $(x^1_0, x^2_0) \in \text{III}$, $0 \longrightarrow -M$,
4) if $(x^1_0, x^2_0) \in \text{IV}$, $M \longrightarrow 0 \longrightarrow -M$.

We shall consider in each case.

1) If $(x^1_0, x^2_0) \in \text{I}$, the phase point starts with $u = 0$. We put the switching time $t_1$. From (15) and (16), we obtain when $0<t<t_1$,

$$x^1(t) = x^2_0 t + x^1_0, \quad x^2(t) = x^2_0, \quad (23)$$

and the first segment of the corresponding trajectory is

$$x^2 = x^2_0, \quad (24)$$

hence, the switching point is $((x^2_0)^2/2M,\ x^2_0)$ and the switching time is

$$t_1 = \frac{x^2_0}{2M} - \frac{x^1_0}{x^2_0}. \quad (25)$$

When $t_1 < t < T$, the phase point moves with $u = M$, therefore

$$x^1 = \frac{1}{2}Mt^2 + c_1 t + c_2, \quad x^2(t) = Mt + c_1, \quad (26)$$

and the last segment of the corresponding trajectory is

$$x^1 = \frac{1}{2M}(x^2)^2. \quad (27)$$

From the continuity conditions of the trajectory, we obtain

$$\frac{1}{2}Mt_1^2 + c_1 t_1 + c_2 = x^2_0 t_1 + x^1_0, \quad Mt_1 + c_1 = x^2_0, \quad (28)$$

and from the terminal conditions,

$$\frac{1}{2}MT^2 + c_1 T + c_2 = 0, \quad MT + c_1 = 0. \quad (29)$$

It follows that

$$T = -\frac{x^2_0}{2M} - \frac{x^1_0}{x^2_0}, \quad (30)$$

and

$$\min J = \int_{t_1}^{T} M dt = -x^2_0. \tag{31}$$

The trajectory corresponding to the optimal control has the form in Fig. 4.

In addition, it is evident that the segment of the straight line $x^2 = x^2_0$ included in I consists of the initial points $(x^1_0, x^2_0) \in$ I from which the phase point moves to the origin with the minimum value J that equals $-x^2_0$ constantly. Moreover, it is an interesting fact that the initial points $(x^1_0, x^2_0)$ which is transferred with the optimal control to the origin in time k exist on the curve(Rf. Fig. 5):

$$x^1 = -\frac{1}{2M} x^2(x^2+2kM), \quad (x^1 \geqq \frac{1}{2M} (x^2)^2). \tag{32}$$

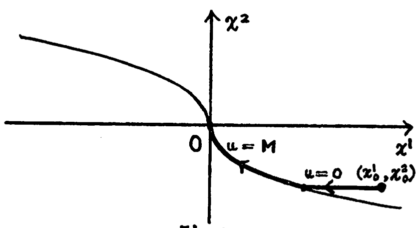

Fig. 4

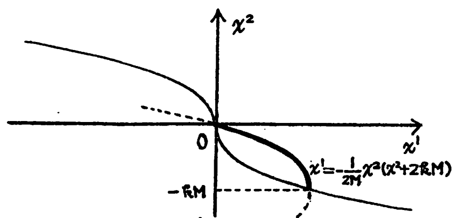

Fig. 5

2) If $(x^1_0, x^2_0) \in$ II, u switches as $-M \longrightarrow 0 \longrightarrow M$. We put the switching times $t_1$ and $t_2$, $(t_1 < t_2)$.

When $0 < t < t_1$, from (17) with initial conditions (4),

$$x^1(t) = -\frac{1}{2} Mt^2 + x^2_0 t + x^1_0, \quad x^2(t) = -Mt + x^2_0, \tag{33}$$

then

$$x^1 = -\frac{1}{2M} \left\{ (x^2)^2 - (x^2_0)^2 \right\} + x^1_0. \tag{34}$$

Similarly when $t_1 < t < t_2$,

$$x^1(t) = c_3 t + c_4, \quad x^2(t) = c_3, \tag{35}$$

i.e.

$$x^2 = c_3, \tag{36}$$

and when $t_2 < t < T$,

$$x^1(t) = \frac{1}{2} Mt^2 + c_1 t + c_2, \quad x^2(t) = Mt + c_1, \tag{37}$$

i.e.

$$x^1 = \frac{1}{2M} \left\{ (x^2)^2 - c_1^2 \right\} + c_2, \tag{38}$$

and from the continuity conditions of the trajectory, we obtain

$$-\frac{1}{2} Mt_1^2 + x^2_0 t_1 + x^1_0 = c_3 t_1 + c_4, \quad -Mt_1 + x^2_0 = c_3, \tag{39}$$

$$\frac{1}{2} Mt_2^2 + c_1 t_2 + c_2 = c_3 t_2 + c_4, \quad Mt_2 + c_1 = c_3, \tag{40}$$

and the terminal conditions of the trajectory take the forms

$$\frac{1}{2}MT^2+c_1T+c_2=0, \quad MT+c_1=0. \tag{41}$$

Moreover, since the switching times are $t_1$ and $t_2$,

$$-at_1+b=\lambda, \quad -at_2+b=-\lambda, \tag{42}$$

and from the maximum value condition we obtain

$$\lambda M+ax^2_0-Mb=0. \tag{43}$$

If there were an optimal process, there would be a unique solution of a, b, $c_1$, $c_2$, $c_3$, $c_4$, $t_1$, $t_2$ and T satisfying the system of nine equations (39), (40), (41), (42) and (43). But the last three equations lead $t_1 = x^2_0/M$ because of existence of a non-trivial solution of (8), and then $c_3 = 0$ from (39). This implies $x^1(t) = c_4 =$ constant with $u(t) = 0$, which contradicts the terminal conditions. It follows that there is not any optimal process.

As a matter of fact, when we suppose that the point (p, q) is the switching point at time $t_1$, there is no optimal control if $q \geqq 0$, and if the point (p, q) is included in region Ⅰ, min J decreases as $q \longrightarrow 0$, on the other hand the time corresponding to the optimal control which transfers the phase point from (p, q) to the origin tends to infinity (Rf. Fig. 5).

3) If $(x^1_0, x^2_0) \in$ Ⅲ, as in the case 1), we obtain the next results. When $0 < t < t_1$, $u = 0$ and the first segment of the trajectory is

$$x^2=x^2_0, \tag{44}$$

and when $t_1 < t < T$, $u = -M$, and then the last segment of the trajectory is

$$x^1=-\frac{1}{2M}(x^2)^2, \tag{45}$$

where

$$t_1=-\frac{x^2_0}{2M}-\frac{x^1_0}{x^2_0}, \tag{46}$$

$$T=\frac{x^2_0}{2M}-\frac{x^1_0}{x^2_0}, \tag{47}$$

therefore

$$\min J=\int_{t_1}^{T} Mdt=x^2_0. \tag{48}$$

the trajectory corresponding to the optimal control has the form shown in Fig. 6.

4) If $(x^1_0, x^2_0) \in$ Ⅳ, there is not any optimal control.

Furthermore, we can find the regions which consist of the initial points transferred with the optimal control to the origin in time less than or equal k. The shaded areas shown in Fig. 7 are the regions considered above.

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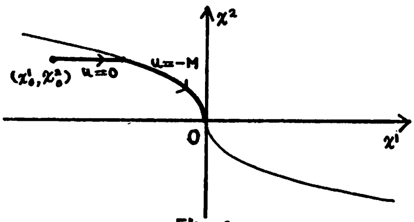


Fig. 6

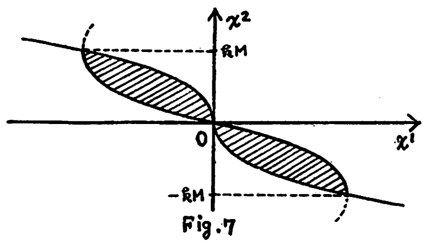


Fig. 7

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