On Lipschitz Uniform Stability of Nonlinear Functional Differential Equations

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1. Introduction

In [1], F.M. Dannan and S. Elaydi introduced the notion of Lipschitz stability for the systems of ordinary differential equations, and made a comparison between Lipschitz stability and Liapunov stability.

For linear systems, the notion of Lipschitz uniform stability and that of Liapunov uniform stability are equivalent. However, for nonlinear systems, the two notions are quite distinct. (cf. [1], [2])

In 1991, Yu-li Fu extends the concept of Lipschitz stability to the systems of functional differential equations. (cf. [3])

In this paper, by using the Liapunov second method and the comparison principle, we will state some extension of the sufficient conditions for Lipschitz uniform stability of the systems of nonlinear functional differential equations.

2. Definitions and Notations

Before giving further details, we give some of the main definitions and notations that we need in the sequel.

Let I and R^+ denote the intervals $[t_0, \infty]$ and $[0, \infty]$ respectively and let R^n denote the Euclidean n-space. We denote $C([a, b], R^n)$ the Banach space of continuous functions mapping the interval $[a, b]$ into R^n with the topology of uniform convergence, and designate the norm of an element $\phi \in C([-r, 0], R^n)$, where $r > 0$, by $\|\phi\| = \sup_{-r \leq \theta \leq 0} |\phi(\theta)|$.

If $\sigma \in I$, $A > 0$ and $x \in C([\sigma - r, \sigma + A], R^n)$, then for any $t \in [\sigma, \sigma + A]$ we let $x_t \in C([-r, 0], R^n)$ be defined by $x_t(\theta) = x(t + \theta)$, $-r \leqq \theta \leqq 0$, i.e., the symbol x_t will denote the restriction of any continuous function $x(u)$ defined on $-r \leqq u < A$, to the interval $[t - r, t]$.

We consider the systems of the functional differential equations

(1) $x'(t) = f(t, x_t)$,

where $x \in R^n$, f: $R \times C([-r, 0], R^n) \to R^n$, $f(t, x_t)$ is continuous and $f(t, 0) \equiv 0$. The initial value condition associated with (1) is

(2) $x(\theta) = \phi(\theta)$, $\theta \in [-r, 0]$, $\phi(\theta) \in C([-r, 0], R^n)$.

We always assume that the solution of (1) with (2) is existent and unique.

(Definition 1.) A function $x(t_0, \phi)$ is said to be a solution of (1) with initial condition $\phi \in C([-r, 0], R^n)$ at $t = t_0$ $t_0 \geqq 0$, if there is an $A > 0$ such that $x(t_0, \phi)$ is a function from $[t_0 - h, t_0 + A)$ into R^n with the properties;

(i) $x_t(t_0, \phi) \in C([-r, 0], R^n)$ for $t_0 \leqq t < t_0 + A$,

(ii) $x_{t_0}(t_0, \phi) = \phi$,

(iii) $x(t_0, \phi)$ satisfies (1) for $t_0 \leqq t < t_0 + A$.

In this paper, we shall denote by $x(t, t_0, \phi)$ the value of $x(t_0, \phi)$ at t.

(Definition 2.) Let $V(t, \phi)$ be a continuous functional defined for $t \geqq 0$, $\phi \in C([-r, 0], R^n)$. The upper right-hand derivative of $V(t, \phi)$ along the solution of (1) will be denoted by $V'_{(1)}(t, \phi)$ and is defined to be

$$V'_{(1)}(t, \phi) = \overline{\lim_{h \to 0+}} \frac{1}{h}\{V(t+h, x_{t+h}(t_0, \phi)) - V(t, \phi)\},$$

where $x(t_0, \phi)$ is the solution of (1) through (t_0, ϕ).

(Definition 3.) For the solution $x(t_0, \phi)$ of (1) through (t_0, ϕ), where $t_0 \geqq 0$, $\phi \in C([-r, 0], R^n)$, if there exists a constant $\delta > 0$, which is independent of t_0, and another constant $M = M(\delta) > 0$, such that

$$\|x_t(t_0, \phi)\| \leqq M\|\phi\| \quad \text{for all } t \geqq t_0 \text{ and } \|\phi\| < \delta,$$

then the zero solution of (1) is said to be Lipschitz uniformly stable.

Next, we consider the systems of ordinary differential equations

(3) $x' = F(t, x)$,

where $x \in R^n$, $F(t, x) \in C(I \times R^n, R^n)$, $F(t, 0) \equiv 0$ and $x(t, t_0, x_0)$ is the solution of (3) with $x(t_0, t_0, x_0) = x_0$, $t_0 \geqq 0$.

Further, we consider a scalar differential equation

(4) $u' = g(t, u)$,

where $g(t, u) \in C(I \times R^+, R)$, $g(t, 0) \equiv 0$ and $u(t, t_0, u_0)$ is the maximal solution of (4) with $u(t_0, t_0, u_0) = u_0$.

(Definition 4.) The zero solution of (3) is said to be Lipschitz uniformly stable if for any $t_0 \geqq 0$, there exist $\delta > 0$ and $M > 0$ such that $\|x(t, t_0, x_0)\| \leqq M\|x_0\|$ for any $\|x_0\| < \delta$ and all $t \geqq t_0$.

(Definition 5.) The zero solution of (4) is said to be Lipschitz uniformly stable if for any $t_0 \geqq 0$, there exist $\delta > 0$ and $M > 0$ such that $u(t, t_0, u_0) \leqq Mu_0$ for all $u_0 < \delta$ and all $t \geqq t_0$.

(Definition 6.) Coresponding to the function $V(t, x) \in C(R^+ \times R^n, R)$, we define the function

$$V'_{(3)}(t, x) = \overline{\lim_{h \to 0+}} \frac{1}{h}\{V(t+h, x+hF(t, x)) - V(t, x)\}.$$

(Definition 7.) The zero solution $x = 0$ of (3) is uniformly stable, if for any $\varepsilon > 0$ and any $t_0 \geqq 0$, there exists a $\delta(\varepsilon) > 0$ such that $\|x_0\| < \delta(\varepsilon)$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geqq t_0$, where $x(t, t_0, x_0)$ denotes the solution of (3) through the point (t_0, x_0).

3. Preliminary Results

In [3], the sufficient condition for Lipschitz uniform stability of functional differential equations (1) was given by Yu-li Fu as follows.

[Theorem 3.1] Assume that there exist a continuous function $g(t, x) \in C(I \times R^1, R^1)$ and a continuous functional $V(t, \phi)$ defined on $I \times C([-r, 0], R^n)$, for which

(i) $V'_{(1)}(t, x_t) \leqq g(t, V(t, x_t))$, for all $t \geqq t_0$,

(ii) $a(\|\phi\|) \leqq V(t, \phi) \leqq b(\|\phi\|)$ for any $\phi \in C([-r, 0], R^n)$, where $a(s)$ and $b(s)$ are continuous and nondecreasing nonzero functions for $s > 0$, satisfying $a(0) \equiv 0$, $b(0) \equiv 0$ and $V(t, 0) \equiv 0$.

If the zero solution of a scalar differential equation (4) is Liapunov uniformly stable, then the zero solution of (1) is Lipschitz uniformly stable.

For the proof of this theorem, see [3].

In 1991, M. Kudo showed the following result with respect to Lipschitz uniform stability of nonlinear ordinary differential equations. (cf. [13])

[Theorem 3.2] Suppose that there exist functions $V(t, x) \in C(I \times R^n, R^+)$, $a(t, r) \in C(I \times R^+, R^+)$, $c(t, r) \in C(I \times R^+, R^+)$ and $g(t, u) \in C(I \times R^+, R)$ such that

(i) $V(t, x)$ is locally Lipschitz in x and $V(t, 0) \equiv 0$,

(ii) $a(t, \|x\|) \leqq V(t, x) \leqq c(t, \|x\|)$, where $a(t, r)$ increases monotonically with respect to t for each fixed r, $a(t, 0) \equiv 0$, $a(t, r) > 0$ for $r \neq 0$,
$kc(t, r) \leqq c(t, kr)$ for a positive constant k and if $a(t, r) \leqq c(t, s)$, then $r \leqq s$,

(iii) $V'_{(3)}(t, x) \leqq g(t, V(t, x))$.

If the zero solution of (4) is Lipschitz uniformly stable, then the zero solution of (3) is also Lipschitz uniformly stable.

For proof, see [13].

4. Main Results

[Theorem 4.1] Suppose that there exist a continuous functional $V(t, \phi)$ defined on $I \times C([-r, 0], R^n)$ and a continuous function $g(t, u) \in C(I \times R^+, R)$ satisfying the following conditions:

(i) $V'_{(1)}(t, x_t) \leqq g(t, V(t, x_t))$ for all $t \geqq t_0 \geqq 0$, $V(t, 0) \equiv 0$,

(ii) $a(t, \|\phi\|) \leqq V(t, \phi) \leqq b(\|\phi\|)$ for all $\phi \in C([-r, 0], R^n)$, where $a(t, r)$ is continuous in (t, r), nondecreasing in r for each fixed t, nondecreasing in t for each fixed r, $a(t, r) > 0$ for all $r \neq 0$ and $a(t, 0) \equiv 0$, $b(r)$ is continuous, nondecreasing, $b(r) > 0$ for all $r \neq 0$ and $b(0) \equiv 0$.

If the zero solution $u = 0$ of the scalar differential equation (4) is Liapunov uniformly stable, then the zero solution of (1) is Lipschitz uniformly stable.

(Proof.) Since the zero solution $u = 0$ of (4) is Liapunov uniformly stable, for any $\varepsilon > 0$ and any $t_0 \geqq 0$, there exists $\delta(\varepsilon) > 0$ such that $u_0 < \delta(\varepsilon)$ implies $u(t, t_0, u_0) < a(0, \varepsilon)$ for all $t \geqq t_0$. Taking $u_0 = b(\|\phi\|)$ for any $\phi \in C([-r, 0], R^n)$ such that $\frac{\varepsilon}{M} < \|\phi\| < b^{-1}(\delta(\varepsilon))$, where $M \geqq 1$ is constant, we find that $V(t_0, \phi) \leqq b(\|\phi\|) = u_0$.

Using the comparison principle, we have that, if $V(t_0, \phi) \leqq u_0$, then $V(t, x_t) \leqq u(t, t_0, u_0)$ for all $t \geqq t_0$, where $u(t, t_0, u_0)$ is the maximal solution of (4) satisfying $u(t_0, t_0, u_0) = u_0$.

Thus, by the condition (ii), we have

$a(0, \|x_t\|) \leqq a(t, \|x_t\|) \leqq V(t, x_t) \leqq u(t, t_0, u_0) < a(0, \varepsilon) < a(0, M\|\phi\|)$.

Then, since $a(t, r)$ is nondecreasing in r for each fixed t, we get $x_t(t_0, \phi) < M\|\phi\|$ for all $t \geqq t_0$ and any $\|\phi\| < \delta(\varepsilon)$. The proof is complete.

[Theorem 4.2] Assume that there exist a continuous functional $V(t, \phi)$ defined on $I \times C([-r, 0], R^n)$ and a continuous function $g(t, u) \in C(I \times R^+, R)$ satisfying the following conditions:

(i) $V'_{(1)}(t, x_t) \leqq g(t, V(t, x_t))$ for all $t \geqq t_0 \geqq 0$, $V(t, 0) \equiv 0$,

(ii) $a(t, \|\phi\|) \leqq V(t, \phi) \leqq b(t, \|\phi\|)$ for all $t \geqq t_0$ and any $\phi \in C([-r, 0], R^n)$, where $a(t, r)$ is continuous in (t, r), nondecreasing in t for each fixed r, $a(t, r) > 0$ for all $r \neq 0$ and $a(t, 0) \equiv 0$, $b(t, r)$ is continous in (t, r), $kb(t, r) \leqq b(t, kr)$ for a positive constant k, and if $a(t, r) \leqq b(t, s)$, then $r \leqq s$ for all t.

If the zero solution $u = 0$ of the scalar differential equation (4) is Lipschitz uniformly stable, then the zero solution of (1) is also Lipschitz uniformly stable.

(Proof.) Since the zero solution $u = 0$ of (4) is Lipschitz uniformly stable, for any $t_0 \geqq 0$, there exist $\delta > 0$ and $M \geqq 1$ such that $u(t, t_0, u_0) \leqq Mu_0$ whenever $u_0 < \delta$.

We put $u_0 = b(t_0, \|\phi\|)$ for any $\phi \in C([-r, 0], R^n)$ such that $\|\phi\| < \delta$. From the condition (i), the application of the comprison principle shows that $V(t, x_t(t_0, \phi)) \leqq u(t, t_0, u_0)$ for all $t \geqq t_0$. Hence we have, by the condition (ii),

$a(t_0, \|x_t\|) \leqq a(t, \|x_t\|) \leqq V(t, x_t(t_0, \phi)) \leqq u(t, t_0, u_0) \leqq Mu_0 = Mb(t_0, \|\phi\|) \leqq b(t_0, M\|\phi\|)$.

Thus we have, for all $t \geqq t_0$ and any $\|\phi\| < \delta$, $\|x_t(t_0, \phi)\| \leqq M\|\phi\|$, which shows that the zero solution of (1) is Lipschitz uniformly stable.

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