On the Estimate for the Perturbations of Nonlinear Perturbed Systems with Some Stability Property

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1. Introduction

Let R^n denote Euclidean n-space. And let R^+ and I denote the interval $[0, \infty)$ and $[t_0, \infty)$, respectively, for some positive real number t_0.

Consider a system of nonlinear differential equations

$$x' = f(t,x), \qquad (1)$$

where $f(t,x)$ is a continuous function defined on $I \times R^n$ with values in R^n and $f(t,0)=0$. If we denote by $C[A,B]$ the set of all continuous functions defined on A with values in B, then we can write $f(t,x) \in C[I\times R^n, R^n]$.

In addition to (1), we consider a system of perturbed nonlinear differential equations

$$y' = f(t,y) + g(t,y), \qquad (2)$$

where $g(t,y) \in C[I\times R^n, R^n]$ and $g(t,0) = 0$.

F.M.Dannan and S.Elaydi [1] introduced the new notions of Lipschitz stability i. e. uniformly Lipschitz stability, uniformly Lipschitz stability in variation and globally uniformly Lipschitz stability in variation (same as globally uniformly stable in variation).

[Definition 1] The zero solution of (1) is said to be uniformly Lipscitz stable, if there exist $M \geqq 1$ and $\delta > 0$ such that $\| x(t,t_0,x_0)\| \leqq M\| x_0\|$ for $\| x_0\| < \delta$ and $t \geqq t_0 > 0$, where $x(t)=x(t,t_0,x_0)$ is the solution of (1) with $x(t_0)=x_0$ and $\| x\|$ is the suitable norm of $x \in R^n$.

The notion of uniformly Lipschitz stability lies between asymptotic stability in variation and uniform stability [1].

Generally, explicit solutions of differential equations are wholly out of the questions.

Liapunov's so-called "second method" has been recognized to be very general and powerful in the qualitative theory of ordinary differential equations [6], [7]. In this method, we consider a continuous scalar function $V(t,x)$ called Liapunov function, and assume some conditions for this function and its derivative along the solution of considering system of differential equations.

[Definition 2] Corresponding to $V(t,x) \in C[R^+\times R^n, R]$, we define the function

$$V'_{(1)}(t,x) = \limsup_{h\to+0} \frac{1}{h}\{V(t+h, x+hf(t,x)) - V(t,x)\}$$

and

$$V'(t,x(t)) = \limsup_{h\to+0} \frac{1}{h}\{V(t+h, x(t+h)) - V(t,x(t))\}.$$

If $V(t,x)$ is locally Lipshitz in x, then we see that

$$V'_{(1)}(t,x) = V'(t,x(t)).$$

In case $V(t,x)$ has continuous partial derivatives of first order, it is evident that

$$V'_{(1)}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t,x)$$

where " • " denotes the scalar product.

In this paper, using this second method, we describe some results with respect to uniformly Lipschitz stability of (1) and (2). And we introduce a new notion similar to uniformly Lipschitz stability and report some results concerning with the estimate for the perturbations of the nonlinear perturbed systems.

2. Preliminary Results

Suppose that there exists a scalar differential eqation

$$u' = r(t,u) \qquad (3)$$

where $r(t,u) \in C[I \times R^+, R]$, $r(t,0) = 0$ and $u(t) = u(t,t_o,u_o)$ is the maximal solution of (2) with $u(t_o,t_o,u_o) = u_o$.

[Definition 3] The zero solution of (3) is said to be uniformly Lipschitz stable, if there exist $M \geqq 1$ and $\delta > 0$ such that $u(t,t_o,x_o) \leqq Mu_o$ for $u_o < \delta$ and $t \geqq t_o > 0$.

The so-called comparison method studies the relationship which should exist between (1) and (3) in order that the stability properties of (3) entail the corresponding properties for (1). First, we give the comparison theorem which is the simplest form of a very general comparison principle.

【Theorem 1】 Suppose that the maximal solution $u(t)$ of (3) such that $u(t_o) = u_o$, stays on closed interval $[\alpha, \beta]$. If continuous function $m(t)$ with $m(t_o) \leqq u_o$ satisfies

$$m'(t) \leqq r(t, m(t)),$$

where $r(t,u)$ is continuous on an open connected set $\Omega \subset R^2$, then we have

$$m(t) \leqq u(t) \text{ for } \alpha \leqq t \leqq \beta.$$

For the proof of this theorem, see references [3], [4] or [5].

Next, we give some results obtained by F.M.Damman and S.Elaydi [2].

Let $S(\delta)$ be the set of $x \in R^n$ such that $\|x\| < \delta$.

【Theorem 2】 Suppose that there exist two functions $V(t,x)$ and $r(t,u)$ satisfying the following conditions :

(i) $r(t,u) \in C[I \times R^+, R]$ and $r(t,0) = 0$,

(ii) $V(t,x) \in C[I \times S(\delta), R^+]$, $V(t,0) = 0$,
$V(t,x)$ is locally Lipschitz in x and satisfies $V(t,x) \geqq b(\|x\|)$, where $b(s) \in C[[0,\delta], R^+]$, $b(0) = 0$, and $b(s)$ is strictly monotone increasing in s such that $b^{-1}(\theta s) \leqq s q(\theta)$ for some function q, with $q(\theta) \geqq 1$ if $\theta \geqq 1$,

(iii) $V'_{(1)}(t,x) \leqq r(t, V(t,x))$ for $(t,x) \in I \times S(\delta)$.

If the zero solution of (3) is uniformly Lipschitz stable, then so is the zero solution of (1).
Moreover, if (ii) is replaced by

(iv) $(\lambda_1(t)\|x\|)^2 \leqq V(t,x) \leqq (\lambda_2(t)\|x\|)^2$,

where $\lambda_1(t)$ and $\lambda_2(t)$ are positive continuous functions with $\lambda_2(t_o) \geqq \lambda_1(t_o)$, then we have the same conclusion.

For the proof of this theorem, see reference [2].

【Theorem 3】 Suppose that the zero solution of (1) is uniformly Lipschitz stable with a Lipschitz constant M and there exists a function $r(t,u) \in C[I \times R^+, R^+]$, $r(t,0) = 0$, which is monotone nondecreasing in u for each $t \in R^+$ and such that

$$\| g(t,y) \| \leqq r(t, \| y \|).$$

If the zero solution of the scalar equation

$$u' = Mr(t,u),\ u(t_0) = u_0 \geqq 0$$

is uniformly Lipshitz stable, then so is the zero solution of (2).

For the proof of this theorem, see reference [2].

3. Uniformly Lipschitz Stability

In the beginning, we describe a result concerning with uniformly Lipschitz stability of a system of differential equations (1), which is obtained formerly.

【Theorem 4】 Suppose that there exist functions $V(t,x) \in C[I \times R^n, R^+]$, $c(t,s) \in C[I \times R^+, R^+]$ and $r(t,u) \in C[I \times R^+, R]$ such that

(i) $V(t,x)$ is locally Lipschitz in x and $V(t,0) = 0$,

(ii) $a(t, \| x \|) \leqq V(t,x) \leqq c(t, \| x \|)$,

where $a(t,s)$ increases monotonically with respect to t for each fixed s, $a(t,0) = 0$, $a(t,s) > 0$ for $s \neq 0$, $kc(t,s) \leqq c(t,ks)$ for a positive constant k and if $a(t,s_1) \leqq c(t,s_2)$, then $s_1 \leqq s_2$,

(iii) $V'_{(1)}(t,x) \leqq r(t, V(t,x))$.

If the zero solution of (3) is uniformly Lipschitz stable, then so is the zero solution of (1).

For the proof of this theorem, see reference [10].

Secondely, we give a result concerning with the estimate of the perturbed turm $g(t,y)$ in order that the zero solution of perturbed systems (2) is uniformly Lipschitz stable.

【Theorem 5】 Suppose that there exists function $V(t,x) \in C[I \times R^n, R^+]$ such that

(i) $V(t,x)$ is locally Lipschitz in x with Lipschitz constant L and $V(t,0) = 0$,

(ii) $\| x \| \leqq V(t,x)$

(iii) $V'_{(1)}(t,x) \leqq 0$.

If the zero solution of the scalar differential equation

$$u' = Lr(t,u) \qquad (4)$$

is uniformly Lipschitz stable, where $r(t,u) \in C[I \times R^+, R^+]$, then so is the zero solution of the perturbed system (2) for $g(t,y)$ such that

$$\| g(t,y) \| < r(t, V(t,y)). \qquad (5)$$

(Proof) From uniformly Lipschitz stability of the zero solution of (4), there exist $\delta > 0$ and some constant $M \geqq 1$ such that

$$u(t,t_0,w_0) \leqq Mw_0 \quad \text{whenever} \quad w_0 < \delta, \qquad (6)$$

where $u(t,t_0,w_0)$ is the solution of (4) with $u(t_0,t_0,w_0) = w_0$.

We put $\delta / L = \xi$. For any y_0 such that $\| y_0 \| < \xi$, if we put

$$L \| y_0 \| = u_0, \qquad (7)$$

we have $u_0 < \delta$. Therefore, from (6), we have

$$u(t,t_0,u_0) \leqq Mu_0. \qquad (8)$$

On the other hand, we obtain $V(t_0,y_0) \leqq L \| y_0 \| = u_0$, because $V(t,x)$ is locally Lipschitz.

Let $y(t) = y(t,t_0,y_0)$ be the solution of (2) with $y(t_0, t_0,y_0) = y_0$. Then the conditions (i), (ii) and (iii) lead the following inequalities :

$V'_{(2)}(t,y) \leqq L \| g(t,y) \| + V'_{(1)}(t,y) \leqq L \| g(t,y) \| \leqq Lr(t,V(t,y))$

Using the comparison theorem, we have

$$V(t,y) \leqq u(t,t_0,u_0). \qquad (9)$$

If we put $ML = K$, by (7), (8) and (9), we have

$$V(t,y(t)) \leqq Mu_0 = ML \| y_0 \| = K \| y_0 \| . \qquad (10)$$

Therefore, from the condition (ii), we have

$$\| y(t) \| \leqq V(t,y(t)) \leqq K \| y_0 \| .$$

Thus, we have $\| y(t,t_0,y_0) \| \leqq K \| y_0 \|$ which shows that the zero solution of (2) is uniformly Lipschitz stable. This completes the proof.

4. Property ULLS

We introduce the following definition.

[Definition 4] We say that the zero solution of (1) has the property ULLS, if there exist $\delta > 0$ and a continuous and monotone increasing function $\phi(s)$ such that

$$\| x(t,t_0,x_0) \| \leqq \phi(\| x_0 \|)$$

for $\| x_0 \| < \delta$ and $t \geqq t_0 \geqq \alpha > 0$, where $x(t,t_0,x_0) = x(t)$ is the solution of (1) with $x(t_0,t_0,x_0) = x_0$ and α is a positive constant.

In this definition 4, the case of $\phi(s) = Ms$ is the definition of uniformly Lipschitz stability.

The property ULLS of the zero solution of (3) is defined similarly.

We now give simple examples for scalar differential equations.

[Example] Consider the two scalar equations,

$$\frac{du}{dt} = \frac{u}{t^2+1} \quad \text{and} \quad \frac{du}{dt} = \frac{\sqrt[3]{u^2}}{t^2+1} .$$

As the solution of the former equation through (t_0,u_0) is $u(t,t_0,u_0) = u_0 \exp(\tan^{-1} t - \tan^{-1} t_0)$, we see that $u(t,t_0,u_0) < u_0 \exp(\pi/2)$. Therefore, if we put $M = \exp(\pi/2)$, then we have $u(t,t_0,u_0) < Mu_0$ and the zero solution of the former equation is uniformly Lipschtz stable.

On the other hard, the solution of the latter equation through (t_0,u_0) is given by $u(t,t_0,u_0) = (3\sqrt[3]{u_0} + \tan^{-1} t - \tan^{-1} t_0)^3/27$. As $u(t,t_0,u_0) < (\sqrt[3]{u_0} + \pi/6)^3$, if we put $\phi(u_0) = (\sqrt[3]{u_0} + \pi/6)^3$, then this function $\phi(s)$ is continuous monotone increasing and $u(t,t_0,u_0) < \phi(u_0)$.

Therefore, the zero solution of the latter equation has the property ULLS. In these cases, as δ is arbitrary, these properties are global.

【Theorem 6】 Suppose that there exist functions $a(t) \in C[R^+,R^+]$, $b(s) \in C[R^+,R^+]$ and $V(t,x) \in C[I \times R^n, R^+]$ such that

(i) $V(t,x)$ is locally Lipschitz in x and $V(t,0) = 0$,

(ii) $a(t)\,b(\| x \|) \leqq V(t,x)$ where the functions $a(t)$ and $b(s)$ are monotone increasing,

(iii) $V'_{(1)}(t,x) \leqq r(t,V(t,x))$.

If the zero solution of the scalar differential equation

$$u' = r(t,u) \qquad (11)$$

has the property ULLS, the zero solution of (1) also has the property ULLS.

(Proof) As the zero solution of (11) has the property ULLS, there exist a positive number δ

and a continuous and monotone increasing function ϕ (s) such that

$$u(t,t_o,w_o) < \phi(w_o) \text{ for any } u_o < \delta \text{ and any } t \geqq t_o \geqq \alpha > 0,$$

where $u(t,t_o,w_o)$ is the solution of (11) with $u(t_o,t_o,w_o) = w_o$ and α is a constant.

If we put $\delta / L = \xi$ for the Lipschitz constant L of V, then we have

$$V(t_o,x_o) \leqq L \| x_o \| < \delta$$

for any x_o such that $\| x_o \| < \xi$. Therefore, if we put $V(t_o,x_o) = u_o$, we have $u_o < \delta$ and

$$u(t,t_o,u_o) \leqq \phi(u_o).$$

Using comparison theorem 1, from the assumption (iii), we have

$$V(t,x(t)) \leqq u(t,t_o,u_o).$$

Hence, by the assumption (ii), we have

$$a(t)\, b(\| x(t) \|) \leqq \phi(u_o) = \phi(V(t_o,x_o)) \leqq \phi(L \| x_o \|).$$

As the function a (t) is monotone increasing, we have $0 < a(\alpha) \leqq a(t_o) \leqq a(t)$ for $0 < \alpha \leqq t_o \leqq t$. And as the function b(s) is monotone increasing, there exists a inverse function b^{-1} which is also monotone increasing. Therefore, we have

$$\| x(t) \| \leqq b^{-1} \left(\frac{\phi(L \| x_o \|)}{a(\alpha)} \right) \equiv \Phi(\| x_o \|).$$

This function $\Phi(s)$ is monotone increasing. Therefore, the zero solution of (1) has the property ULLS. This completes the proof.

(Remark) From the proof above, it can be easily seen that the condition (ii) in Theorem 6 is replaced by the condition

(ii)' $a(t, \| x \|) \leqq V(t,x)$, where $a(t,s) \in C[R^+ \times R^+, R^+]$ and increases monotonically with respect to t for each fixed s and s for each fixed t.

Lastly, we describe a result concerning with the estimate of the perturbed term g(t,y) in order that the zero solution of the perturbed systems (2) has the property ULLS.

【Theorem 7】 Suppose that there exist functions $a(t,s) \in C[R^+ \times R^+, R^+]$ and $V \in C[I \times R^n, R^+]$ such that

(i) V(t,x) is locally Lipschitz in x with Lipschitz constant L and V(t,0) = 0,

(ii) $a(t, \| x \|) \leqq V(t,x)$, where a(t,s) increases monotonically with respect to t for each fixed s and s for each fixed t,

(iii) $V'_{(1)}(t,x) \leqq 0$.

If the zero solution of the scalar differential equation

$$u' = Lr(t,u) \tag{12}$$

has the property ULLS, where $r(t,u) \in C[I \times R^+, R^+]$, the zero solution of the perturbed system (2) also has the property ULLS for g(t,x) such that

$$\| g(t,x) \| \leqq Lr(t,V(t,y)). \tag{13}$$

(Proof) As the zero solution of (12) has the property ULLS, there exist a positive number δ and a continuous and monotone increasing function ϕ (s) such that

$$u(t,t_o,u_o) < \phi(u_o) \tag{14}$$

for any $u_o < \delta$ and any $t \geqq t_o \geqq \alpha > 0$, where $u(t,t_o,u_o)$ is the solution of (12) with $u(t_o,t_o,u_o) = u_o$ and α is a constant.

If we put $\delta / L = \xi$, then we have from (i)

$$V(t_o,y_o) \leqq L \| y_o \| < \delta \tag{15}$$

for any y_0 such that $\| y_0 \| < \xi$. Therefore, if we put $V(t_0,y_0) = u_0$, we have $u_0 < \delta$ and the inequality (14) is satisfied.

Let $y(t) = y(t,t_0,y_0)$ be the solution of (2) with $y(t_0,t_0,y_0) = y_0$. Then the condition (iii) and (13) lead the inequality

$$V'_{(2)}(t,y) \leqq Lr(t,V(t,y)).$$

Using comparison theorem, we have

$$V(t,y) \leqq u(t,t_0,u_0). \qquad (16)$$

From (14), (15), (16) and the condition (ii), we have

$$a(\alpha, \| y(t) \|) < a(t, \| y(t) \|) \leqq V(t,y(t)) \leqq u(t,t_0,u_0) \leqq \phi(u_0)$$
$$= \phi(V(t_0,y_0)) \leqq \phi(L \| y_0 \|).$$

As the function $a(\alpha,s) \equiv A(s)$ increases monotonically with respect to s, we see that there exists the inverse function A^{-1} of A. Therefore we have,

$$\| y(t) \| \leqq A^{-1}(\phi(L \| y_0 \|)) \equiv \Psi(\| y_0 \|).$$

As the function $\Psi(s)$ is monotone increasing, the zero solution of (2) has the property ULLS. This completes the proof.

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