# On the Uniformly Lipschitz Stability of Nonlinear Differential Equations by the Comparison Principle

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(Received on 31 October, 1990)

## 1. Introduction

F.M.Dannan and S.Elaydi introduced the new notions of Lipschitz stability i.e. uniformly Lipschitz stability, globally uniformly Lipschitz stability, uniformly Lipschitz stability in variation and globally uniformly Lipschitz stability in variation (same as globally uniformly stable in variation). They showed that these notions are equivalent to the old notions of stability i.e. uniform stability in linear systems but not in nonlinear systems. Furthermore,criteria are given for the notions and several examples are given to demonstrate the differences among the new and the old notions of stability.

They pursued this study in [2]. In it's paper, the techniques of Liapunov functions are used in order to have sufficient conditions for the Lipschitz stability.

In this paper, we describe a result considering the uniformly Lipschitz stability of nonlinear differential equations by using the compaison principle.

## 2. Definitions and Notations

Let I and $R^+$ denote the intervals $[t_0,\infty)$ and $[0,\infty)$ respectively. And let $R^n$ denote Euclidean n-space.

For $x\in R^n$, let the norm of x be $\|x\|$.

We shall denote by C[A,B] the set of all continuous functions defined on A with values in B. Consider the differential system

$$x'=f(t,x), \qquad (1)$$

where $f\in C[I\times R^n,R^n]$, $f(t,0)=0$ and $x(t,t_0,x_0)\equiv x_0(t)$ is the solution of (1) with $x(t_0,t_0,x_0)=x_0$, where $t_0\geqq 0$.

Further, we consider a scalar differential equation

$$u'=g(t,u), \qquad (2)$$

where $g\in[I\times R^+,R]$, $g(t,0)=0$ and $u(t,t_0,u_0)\equiv u(t)$ is the maximal solution of (2) with $u(t_0,t_0,u_0)=u_0$.

Definition 1.[1] The zero solution of (1) is said to be uniformly Lipschitz stable if there exist $M\geqq 1$ and $\delta>0$ such that $\|x(t_0,t_0,u_0)\|\leqq M\|x_0\|$ for $\|x_0\|<\delta$ and $t\geqq t_0\geqq 0$.

Uniformly Lipschitz stability of the zero solution of (2) is defined similarly.

Definition 2. The zero solution of (2) is said to be uniformly Lipschitz stable if there exist $M\geqq 1$ and $\delta>0$ such that $u(t_0,t_0,u_0)\leqq Mu_0$ for $u_0<\delta$ and $t\geqq t_0\geqq 0$.

Definition 3. Corresponding to $V \in C[R^+ \times R^n, R]$ we define the functions

$$V'_{(1)}(t,x) = \sup_{h \to +0} \lim \frac{1}{h}\{V(t+h, x+hf(t,x)) - V(t,x)\}$$

and

$$V'(t,x) = \sup_{h \to +0} \lim \frac{1}{h}\{V(t+h, x(t,+x)) - V(t,x)\}.$$

If V is locally Lipschtz with respect to x, then $V'_{(1)}(t,x) = V(t,x)$.

In case V(t,x) has continuous partial derivatives of first order, it is evident that

$$V'_{(1)}(t,x) = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \cdot f(t,x)$$

where "•" denotes the scalar product.

## 3. Preliminary results

【Theorem 1】 Suppose that the maximal solution u(t) of (2) such that $u(t_0) = u_0$, stays on interval [a,b].

If a continuous function x(t) with $x(t_0) \leqq u_0$ satisfies

$x'(t) \leqq g(t, x(t))$,

where g(t,u) is continuous on an open connected set $\Omega \subset R^2$, then we have

$x(t) \leqq u(t)$ for $a \leqq t \leqq b$.

For the proof of this theorem, see references [3], [4]or[5].

Let $S(\delta) = \{x \in R^n : |x| < \delta\}$.

【Theorem 2】 Let $g \in C[I \times R^+, R]$, $g(t,0) = 0$, such that $|g(t,u) - g(t,v)| \leqq L|u-v|$, for some positive constant L.

Suppose also that

$$\|x + hf(t,x)\| \leqq \|x\| + hg(t, \|x\|) + \varepsilon(h),$$

For $(t,x) \in I \times S(p)$ and for all sufficiently small $h > 0$, with $\lim_{h \to 0} [\varepsilon(h)/h] = 0$.

Then, if the zero solution of the scalar equation (2) is uniformly Lipschitz stable, then the zero solution of (1) uniformly Lipschitz stable.

For the proof of this theorem, see reference [1].

【Theorem 3】 Suppose that there exist two functions V(t,x) and g(t,u) satisfying the following conditions:

(i) $g(t,u) \in C[I \times R^+, R]$ and $g(t,0) = 0$,

(ii) $V(t,x) \in C[I \times S(\delta), R^+]$, $V(t,0) = 0$, V(t,x) is locally Lipschitz in x and satisfies $V(t,x) \geqq b(\|x\|)$, where $b(r) \in C[[0,\delta], R^+]$,

b(0)=0, and b(r) is strictly monotone increasing in r such that

$b^{-1}(\alpha r) \leqq rq(\alpha)$ for some function q,

with $q(\alpha) \geqq 1$ if $\alpha \geqq 1$.

(iii) For $(t,x) \in I \times S(\delta)$, $V'_{(1)}(t,x) \leqq g(t, V(t,x))$.

If the zero solution of (2) is uniformly Lipschitz stable, then so is the zero solution of (1).

For the proof of this theorem, see reference [2].

It is described in [2] that the above theorem remains varid if (ii) is replaced by

(ii)' $(\lambda_1(t)\,|\,x\,|\,)^2 \leqq V(t,x) \leqq \lambda_2(t)\,|\,x\,|\,)^2$, where $\lambda_1(t)$ and $\lambda_2(t)$ are positive continuous functions with $\lambda_2(t_0) \geqq \lambda_1(t)$.

## 4. Main result

【Theorem 4】 Suppose that there exist functions $V \in C[I \times R^n, R^+]$, $a \in C[I \times R^+, R^+]$, $c \in C[I \times R^+, R^+]$
and $g \in C[I \times R^+, R]$ such that

(i) $V(t,x)$ is locally Lipschitz in x and $V(t,0)=0$,

(ii) $a(t, \|x\|) \leqq V(t,x) \leqq c(t, \|x\|)$,

where $a(t,r)$ increases monotonically with respect to t for each fixed r, $a(t,0)=0$, $a(t,r)>0$ for $r \neq 0$, $kc(t,s) \leqq c(t,ks)$ for a positive constant k and if $a(t,r) \leqq c(t,s)$, then $r \leqq s$,

(iii) $V'_{(1)}(t,x) \leqq g(t, V(t,x))$.

If the zero solution of (2) is uniformly Lipschitz stable, then so is the solution of (1).

Proof. From the uniformly Lipschitz stability of the zero solution $u=0$ of (2), there exist $\delta>0$ and some constant $M \geqq 1$ such that

$u(t,t_0,u_0) \leqq Mu_0$, whenever $u_0 < \delta$.

Let L be the Lipschitz constant with respect to x.

If we put $\delta / L = \delta'$, then we have

$$V(t_0,u_0) \leqq L\|x_0\| = \delta$$

for any $x_0$ such that $\|x_0\| \leqq \delta'$.

Therefore if we put $V(t_0,u_0)=u_0$, we have $u_0 < \delta$ and $u(t,t_0,u_0) \leqq Mu_0$.

Using the comparison theorem 1, from the condition (iii), we have

$$V(t,x(t)) \leqq u(t,t_0,u_0).$$

Hence, by the condition (ii), we have

$$\begin{aligned} a(t_0, \|x(t,t_0,u_0)\|) \leqq\ & a(t, \|x(t)\|) \leqq V(t,x(t)) \\ & \leqq u(t,t_0,u_0) \leqq Mu_0 \\ & = MV(t_0,u_0) \leqq Mc(t_0, \|x_0\|) \\ & \leqq c(t_0, M\|x_0\|). \end{aligned}$$

Thus we have $\|x(t,t_0,u_0)\| \leqq M\|x_0\|$ which shows that the zero solution of (1) is uniformly Lipschitz stables.

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