On L^p-Integral Stability

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1. Introduction

A variant of the notion of the total stability is obtained if instead of requiring that the permanent perturbation be small all along , we only require that they be small in the mean.

A slightly different variant of the same type of stability , equally based on the idea of considering perturbations which can be great in certain moments but are small in the mean ,has been defined by Ivo Vrkoc in 1959 . It is called the integral stability .

What further restrictions are necessary and sufficient for any solution of the ordinary differential equations to be integrable on the half-line to the right of the initial point ?

This question was asked by L . Gesari , in 1963 , and was answered by him for the second order linear differential equations 〔12〕. J.J.Levin and J.A.Nohel 〔13〕〔14〕also have obtainned some results in this direction using Liapunov's second method .

To provide a more complete answer to this question , in 1964, A . Strauss first introduced and studied a new kind of a stability , i . e ., L^p-stability of ordinary differential equations ,in his doctoral dissertation 〔15〕.

Many authers have discussed the integral stability and the L^p-stability . (cf . 〔1〕,〔2〕,〔3〕,〔5〕,〔6〕,〔12〕,〔13〕,〔14〕,〔15〕,〔16〕,〔17〕.)

We also have obtained some results of the integral stability and the L^p-stability . (cf . 〔7〕,〔8〕,〔9〕,〔10〕,〔11〕,〔18〕,〔19〕.)

We have combined two notions and introduce a new stability notion . We call this new stability a L^p-integral stability .

As is well known , Liapunov's second method has its origin in three simple theorems that form the core of what he himself called his second method for dealing with questions of stability. It is widely recognized , today , as an indispensabie tool not only in the theory of stability but also in studying many other qualitative properties of solutions of differential equations .

The main characteristic of this method is a introduction of a function , namely the Liapunov function , which defines a generalized distance from the origin of the motion space .

Liapunov's second method is a very useful and powerful instrument in discussing the stability of the solutions of the differential equations . Its power and usefulness lie in the fact that decision is made by investigating the differential equation itself and not by finding solutions of the differential equations . However , it is great difficult to find the Liapunov function satisfying certain conditions . Therefore , it is important to obtain a weak sufficient condition for a stability theorem .

C.Corduneanu 〔20〕 and H.A.Antosiewicz 〔21〕observed that the Liapunv's second method depends basically on the fact that a function $u(t)$ satisfying the inequality $u' \leq g(t,u(t))$, $u(t_0) \leq r_0$ is majorized by the maximal solution of the scalar differential equation $r'=g(t,r)$, $r(t_0) = r_0$. As the comparison principle reduces the problem of determining the behavior of the solution of the differential equations to the solution of the scalar differential equation , it is a very important tool in applications . It is particularly useful in dearing with a variety of qualitative problem .

In this paper , by using the Liapunov's second method and the comparison principle , we will state some sufficient conditions for the L^p-integral stability of solutions of the ordinary differential equations .

2. Notations and Definitions

First , we summarize some basic notations and definitions we will need later on .

Let I denote the interval $0 \leqq t < \infty$, R^n denote Euclidean n-space . For $x \varepsilon R^n$, let $\| x \|$ be any norm of x and shall denote by S_H the set of x suxh that $\| x \| < H, H > 0$.

We shall denote by $C(I \times R^n, R^n)$ the set of all continuous functions defined on $I \times R^n$ valued in R^n.

We consider the system of differential equations

(1) $$\frac{dx}{dt} = f(t,x) \ , \ f(t,0) \equiv 0, \text{ where } f(t,x) \varepsilon C(I \times R^n, R^n),$$

and its perturbed system

(2) $$\frac{dx}{dt} = f(t,x) + F(t,x), \text{ where } F(t,x) \varepsilon C(I \times R^n, R^n).$$

Suppose that $f(t,x)$ and $F(t,x)$ are smooth enough to ensure existence , uniqueness and continuous dependence of the solutions of the initial value problem associated with (1),(2). Furthermore , consider a scalar differential equation

(3) $$\frac{du}{dt} = g(t,u), \ g(t,0), \equiv 0, \text{ where } g(t,u) \varepsilon C(I \times R, R),$$

and its perturbed equation

(4) $$\frac{du}{dt} = g(t,u) + G(t,u), \text{ where } G(t,u) \varepsilon C(I \times R, R).$$

Suppose that $g(t,u)$ and $G(t,u)$ are smooth enough to ensure existence , uniqueness and continuous dependence of the solutions of the initial value problem associated with (3),(4).

Throughout this paper , a solution through a point $(t_0, x_0) \varepsilon I \times R^n$ will be denoted by such a form as $x(t, t_0, x_0)$.

We introduce the following definitions .

〔Definition 1〕 Corresponding to a continuous scalar function $V(t,x)$ defined on an open set, we define the function

$$\dot{V}_{(1)}(t,x) = \overline{\lim_{h \to 0+}} \frac{1}{h} \{V(t+h, x+hf(t,x)) - V(t,x)\}.$$

In case $V(t,x)$ has continuous partial derivatives of the first order , it is evident that

$$\dot{V}_{(1)}(t,h) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t,x),$$

where "." denote a scalar product .

〔Definition 2〕 The zero solution of the system (1) is said to be stable if for any $\varepsilon > 0$ and any $t_o \geqq 0$ there exists $\delta(\varepsilon, t_o) > 0$ such that inequality
$\| x_o \| < \delta(\varepsilon, t_o)$ implies $\| x(t, t_o, x_o) \| < \varepsilon$ for all $t \geqq t_o$.

〔Definition 3〕 The zero solution of the system (1) is said to be L^p-integrally stable, where p is a positive integer, if it is stable and for any $\varepsilon > 0$ and any $t_o \geqq 0$ there exist $\delta_1(\varepsilon, t_o) > 0$ and $\delta_2(\varepsilon, t_o) > 0$ such that $\| x_o \| < \delta_1(\varepsilon, t_o)$ and

$\int_{t_o}^{\infty} \sup_{\| x \| \leqq \varepsilon} \| F(t,x) \| dt < \delta_2(\varepsilon, t_o)$ implies $\int_{t_o}^{\infty} \| x(t, t_o, x_o) \|^p dt < \infty$, where $x(t, t_o, x_o)$ denotes a solution of a perturbed system (2) satisfying an initial condition $x(t_o, t_o, x_o) = x_o$.

〔Definition 4〕 The zero solution of (3) is said to bo L^1-integrally stable if it is stable and if for any $\varepsilon > 0$ and any $t_o \geqq 0$ there exist $\eta_1(\varepsilon, t_o) > 0$ and $\eta_2(\varepsilon, t_o) > 0$

such that $u_o < \eta_1(\varepsilon, t_o)$ and $\int_{t_o}^{\infty} \sup_{u \leqq \varepsilon} | G(t,u) | dt < \eta_2(\varepsilon, t_o)$ implies $\int_{t_o}^{\infty} u(t, t_o, u_o) dt < \infty$,

where $u(t, t_o, u_o)$ is a solution of a perturbed equation (4) satisfying an initial condition $u(t, t_o, u_o) = u_o$.

3. Result

Before we state main result, we give the following theorems we shall need later on.
〔Theorem 1〕 *The zero solution of the system (1) is L^p-integtally stable if and anly if for any $\varepsilon > 0$ and $t_o \geqq 0$ there exist $\delta_1(\varepsilon, t_o) > 0$ and $\delta_2(\varepsilon, t_o) > 0$ such that if $\phi(t)$ is any continuos function defined on $[\, t_o, \infty \,]$ and satisfies $\int_{t_o}^{\infty} \| \phi(t) \| dt < \delta_2(\varepsilon, t_o)$ then any solution $y(t, t_o, y_o)$ satisfying $\| y_o \| < \delta_1(\varepsilon, t_o)$ of the system*

$$(5) \qquad \frac{dy}{dt} = f(t,y) + \phi(t)$$

verifies the inequality $\int_{t_o}^{\infty} \| y(t, t_o, y_o) \|^p dt < \infty$.

Proof. The necessity of the condition is clear. Therefore it suffices to prove that if the property from the statement occures, the zero solution of the system (1) is L^p-integrally stable.

Let $F(t,x)$ be such that $\int_{t_o}^{\infty} \sup_{\| x \| \leqq \varepsilon} \| F(t,x) \| dt < \delta_2(\varepsilon, t_o)$, where $\delta_2(\varepsilon, t_o)$ is one given the condition. Consider a point x_o with $\| x_o \| < \delta_1(\varepsilon, t_o)$ and the solution $x(t, t_o, x_o)$ of the system (2).

If we would not have $\int_{t_0}^{\infty} \| x(t,t_0,x_0) \|^p dt < \infty$, there exists the first point $t_1 > t_0$

$\int_{t_0}^{t_1} \| x(t,t_0,x_0) \|^p dt = \infty$ and $\int_{t_0}^{t} \| x(t,t_0,x_0) \|^p dt < \infty$ for all $t \in [t_0,t_1)$. For any $t \in [t_0,t_1)$ take $\phi(t) = F(t,t_0,x_0)$, we have

$$\int_{t_0}^{t_1} \| \phi(t) \| dt \leqq \int_{t_0}^{t_1} \sup_{\|x\| \leqq \varepsilon} \| F(t,x) \| dt < \delta_2(\varepsilon,t_0) .$$

We extend $\phi(t)$ continuously on the whole semiaxis $t \geqq t_0$ such that

$\int_{t_0}^{\infty} \| \phi^*(t) \| dt < \delta_2(\varepsilon,t_0)$, where $\phi^*(t)$ is a extended function, for this it is sufficient to take $t_2 > t_1$ such that

$$t_2 - t_1 < \frac{2\left(\delta_2(\varepsilon,t_0) - \int_{t_0}^{t_1} \| \phi(t) \| dt\right)}{1 + \| \phi(t_1) \|} ,$$

$\phi^*(t) = \phi(t)$ for all $t \in [t_0,t_1]$, linear on $[t_1,t_2]$, where we put $\phi^*(t_2) = 0$, and zero for all $t \geqq t_1$.

We consider the following system (6)

$$(6) \qquad \frac{dy}{dt} = f(t,y) + \phi^*(t) .$$

Now let $y(t,t_0,x_0)$ be a solution of the system (6) .

From $\| x_0 \| < \delta_1(\varepsilon,t_0)$ and $\int_{t_0}^{\infty} \phi^*(t)dt < \delta_2(\varepsilon,t_0)$, we have $\int_{t_0}^{\infty} \| y(t,t_0,x_0) \|^p dt < \infty$:

hence $\int_{t_0}^{t_1} \| y(t,t_0,x_0) \|^p dt < \infty$. But we have $y(t,t_0,x_0) \equiv x(t,t_0,x_0)$ on $[t_0,t_1]$,hence

$\int_{t_0}^{t_1} \| x(t,t_0,x_0) \|^p dt < \infty$, which is contradictory . Therefore , the zero solution of the system (1) is L^p-integrally stable .

〔Theorem 2〕 *Assume that there exist function $V(t,x) \varepsilon C(I \times S_H, I)$ and $g(t,u) \varepsilon C(I \times I, R)$, and $g(t,0) \equiv 0$ satisfying the following conditions ;*

(i) *$a(\| x \|) \leqq V(t,x)$, $V(t,0) \equiv 0$, where $a(r)$ is a continuous , increasing and positive definite function ,*

(ii) *$| V(t,x) - V(t,y) | \leqq M \| x-y \|$, where M is a positive constant ,*

(iii) *$\dot{V}_{(1)}(t,x) \leqq g(t, V(t,x))$.*

Then , the stability of the zero solution of the equation (3) implies the stability of the zero solution of the system (1).

For proof , see 〔6〕.

Next , we state a main result .

〔Theorem 3〕 *Suppose that there exist functions* $V(t,x) \in C(I\times S_H, I)$ *and* $g(t,u) \in C(I\times I, R)$, *satisfying the following conditions :*

(i) $A\|x\|^P \leqq V(t,x)$, *where* $A > 0$ *is a constant* , $V(t,0) \equiv 0$,

(ii) $|V(t,x) - V(t,y)| \leqq M\|x-y\|$ *for any* $(t,x), (t,y) \in I\times S_H$, *where* $M>0$ *is a constant* ,

(iii) $\dot{V}_{(1)}(t,x) \leqq g(t, V(t,x))$.

If the zero solution of the equation (3) *is* L^1*-integrally stable , then the zero solution of the system* (1) *is* L^P*-integrally stable .*

Proof. By Theorem 2, the zero solution of the system (1) is stable . Thus , for any $\varepsilon \geqq 0$ and any $t_0 \geqq 0$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that inequality $\|x_0\| < \delta$ implies inequality $\|x(t,t_0,x_0)\| < \varepsilon$ for all $t \geqq t_0$.

Since the zero solution of the equation (3) is L^1-integrally stable, it is stable and for any $\varepsilon > 0$ and any $t_0 \geqq 0$,there exist $\eta_1 = \eta_1(\varepsilon, t_0) > 0$ and $\eta_2 = \eta_2(\varepsilon, t_0)$ such that $u_0 < \eta_1$ and $\int_{t_0}^{\infty} \sup_{u\leqq\varepsilon} |G(t,u)|\,dt < \eta_2$ implies $\int_{t_0}^{\infty} u(t,t_0,u_0)dt < \infty$,where $u(t,t_0,u_0)$ is the solution of the perturbed equation (4).

Because $V(t,x)$ is continuous and $V(t,0) \equiv 0$, for given $\eta_1 > 0$ and $t_0 \geqq 0$ there exists $\delta_0(\varepsilon, \eta_1, t_0) = \delta_1(\varepsilon, t_0) > 0$ satisfying $\|x_0\| < \delta_1$ imlies $V(t_0, x_0) < u_0$, where $0 < u_0 < \eta_1$.

Given $\eta_2 > 0$, there exists $\delta_1 = \delta_2(\varepsilon, t_0) > 0$ such thet $\delta_2 < \min(\eta_2/M, \delta)$. By using the condition (iii) ,

$$\dot{V}_{(2)}(t,x) \leqq \dot{V}_{(1)}(t,x) + M\|F(t,x)\| \leqq g(t,V(t,x)) + M\|F(t,x)\| .$$

We put $\lambda(t) = M\|F(t,x(t,t_0,x_0))\|$,where $x(t,t_0,x_0)$ is the solution of the perturbed system (2) such that $\|x_0\| \leqq \delta_1$.

For this function $\lambda(t)$, we consider the following scalar equation

$$\frac{du}{dt} = g(t,u) + \lambda(t). \tag{7}$$

Because $\|x(t,t_0,x_0)\| < \varepsilon$ for all $t \geqq t_0$ satisfying the condition $\|x_0\| < \delta_1$,

we get $\int_{t_0}^{\infty} \lambda(t)dt = M\int_{t_0}^{\infty} \|F(t,x(t,t_0,x_0))\|dt \leqq M\int_{t_0}^{\infty} \sup_{\|x\|\leqq\varepsilon} \|F(t,t_0,x_0))\|dt$

$$\leqq M\delta_2 < \eta_2.$$

By Theorem 1, we have

$$\int_{t_0}^{\infty} r(t,t_0,u_0)dt < \infty , \tag{8}$$

where $r(t,t_0,u_0)$ is the maximal solution of the equation (7) such that $u_0 < \eta_1$.

Moreover , the application of the comparison principle shows that if $V(t_0,x_0) \leqq u_0$, then

(9) $V(t,x(t,t_0,x_0)) \leqq r(t,t_0,u_0)$ for all $t \geqq t_0$.

For any $\varepsilon > 0$ and any $t_0 \geqq 0$, using δ_1 and δ_2 defined above , we claim that the zero solution of the system (1) is L^P-integrally stable whenever $\|x_0\| < \delta_1$ and

$$\int_{t_0}^{\infty} \sup_{\|x\|\leqq\varepsilon} \|F(t,x)\|dt < \infty.$$

By using the condition (i) and relations (8) and (9), it follows that
$A \| x(t,t_0,x_0) \|^P \leqq V(t,x(t,t_0,x_0)) \leqq r(t,t_0,u_0)$,
and hence

$$\int_{t_0}^{\infty} \| x(t,t_0,x_0) \|^P dt \leqq \frac{1}{A} \int_{t_0}^{\infty} r(t,t_0,u_0) dt < \infty .$$

This proves that the zero solution of the system (1) is L^P-integrally stable .

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平成３年２月

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