On the Partially Uniformly Integral Stability of Solutions of Ordinary Differential Equations

Shoichi SEINO

(Received on 9, October. 1989)

1. Introduction

A variant of the notion of total stability is obtained if inquiring that the permanent perturbations be small all along, .we only require that they be small in the mean. A slightly different variant of the same type of stability, equally based on the idea of considering perturbations which can be great in certain moments but are small in the mean, has been defined by lvo Vrkoc. It is called the integral stability.

Many authors have discussed the integral stability. (cf. [1],[2],[3],[4],[5],[6],[7],[8],[9].)

As is well known, Liapunov's second method has its origin in three simple theorems that form the core of what he called the second method for dealing with questions of stability. It is widely recognized as an indispensable tool not only in the theory of stability but also in studying many other qualitative properties of solutions of differential equations. The main characteristic of this method is the introduction of a function, namely the Liapunov function, which defines a generalized distance from the origin of the motion space.

Liapunov's second method is a very useful and powerful instrument in discussing the stability of the system of differential equations. Its power and usefulness lie in the fact that the decision is made by investigating the differential equation itself and not by finding solutions of the differential equations. However, it is great difficult to find the Liapunov function satisfying certain conditions. Therefore, it is important to obtain a weak sufficient condition for a stability theorem.

In many applications, we need to see the qualities not of the whole solution but of the partial.

In this paper, by using the Liapunov's second method, we will state extension of the sufficient conditions for the partially uniformly integral stability.

2. Notations and Definitions

First, we summarize some basic notations and definitions we will need later on.

Let I denote the interval $0 \leqq t < \infty$, R^n denote Euclidean n-space. For $x \in R^n$, let $\| x \|$ be any norm of x and we shall denote by S_H the set of x such that $\| x \| < H$, $H > 0$.

We consider a system of differential equations

$$(1) \begin{cases} \dfrac{dx}{dt} = f(t,x,y), \ f(t,0,0) = 0, \\ \dfrac{dy}{dt} = g(t,x,y), \ g(t,0,0) = 0, \end{cases}$$

where x is an n-vector, y is an m-vector, f(t,x,y) is an n-vector function and g(t,x,y) is an m-vector function.

Suppose that f(t,x,y) is continuous on $I \times R^n \times R^m$, g(t,x,y) is continuous on $I \times R^n \times R^m$ and that

$f(t,x,y)$ and $g(t,x,y)$ are smooth enough to ensure existence, uniqueness and continuous dependence of the solutions of the initial value problem associated with (1). We shall denote by $C(I\times R^n\times R^m, R^K)$ the set of all continuous functions defined on $I\times R^n\times R^m$ with valued in R^K. Throughout this paper, a solution through a point $(t_0,x_0,y_0)\in I\times R^n\times R^m$ will be denoted by such a form as $(x(t,t_0,x_0,y_0), y(t,t_0,x_0,y_0))$. We introduce the following definitions.

(Definition 2.1) Corresponding to a continuous scalar function $V(t,x,y)$ defined on an open set, we define the function

$$\dot{V}_{(1)}(t,x,y)=\overline{\lim_{h\to o+}}\frac{1}{h}\{V(t+h,x+hf(t,x,y),y+hg(t,x,y))-V(t,x,y)\}$$

In case $V(t,x,y)$ has continuous partial derivatives of the first order, it is evident that

$$\dot{V}_{(1)}(t,x,y)=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial y}\cdot f(t,x,y)+\frac{\partial V}{\partial y}\cdot g(t,x,y),$$

where "$\cdot$" denote a scalar product.

3. Partially Uniformly Integral Stability

We consider the system of differential equations

$$(1)\quad \begin{cases} \dfrac{dx}{dt}=f(t,x,y),\ f(t,0,0)=0,\ \text{where } f(t,x,y)\in C(I\times R^n\times R^m,R^n) \\ \dfrac{dy}{dt}=g(t,x,y),\ g(t,0,0)=0,\ \text{where } g(t,x,y)\in C(I\times R^n\times R^m,R^m), \end{cases}$$

and its perturbed system

$$(2)\quad \begin{cases} \dfrac{dx}{dt}=f(t,x,y)+F(t,x,y),\ \text{where } F(t,x,y)\in C(I\times R^n\times R^m,R^n) \\ \dfrac{dy}{dt}=g(t,x,y)+G(t,x,y),\ \text{where } G(t,x,y)\in C(I\times R^n\times R^m,R^m). \end{cases}$$

Let us first define the notion of the partially integral stability.

(Definition 3.1) The zero solution of the system (1) is said to be partially integrally stable with respect to x if for any $\varepsilon>0$ and any $t_0\geqq 0$ there exist $\delta_1(t_0,\varepsilon)>0$ and $\delta_2(t_0,\varepsilon)>0$ such that $\|x_0\|+\|y_0\|<\delta_1(t_0,\varepsilon)$ and

$$\int_{t_0}^{\infty}\sup_{\|x\|\leq\varepsilon}\{\|F(t,x,y)\|+\|G(t,x,y)\|dt\}<\delta_2(t_0,\varepsilon)$$

implies $\|x(t,t_0,x_0,y_0)\|<\varepsilon$ for all $t\geqq t_0$, where $(x(t,t_0,x_0,y_0),y(t,t_0,x_0,y_0))$ denotes a solution of the system (2) satisfying a initial condition $(x(t_0,t_0,x_0,y_0),y(t_0,t_0,x_0,y_0))=(x_0,y_0)$.

(Definition 3.2) The zero solution of the system (1) is said to be partially uniformly integrally stable with respect to x if δ_1 and δ_2 in Definition 3.1 are independent of t_0.

In [1], A.Halanay proved the next theorem.

[Theorem 3.1] *If there exists a continuous function $V(t,x)$ defined on $I\times S_H$ with properties:*

(i) $a(\|x\|)\leqq V(t,x)$, $V(t,0)=0$, *where $a(r)$ is a continuous, positive definite and increasing function,*

(ii) $|V(t,x)-V(t,y)|\leqq M\|x-y\|$, $M>0$,

(iii) $\dot{V}_{(3)}(t,x)\leqq h(t)V(t,x(t,t_0,x_0))$, *with* $\int_0^{\infty}h(t)dt<\infty$, $h(t)\geqq 0$, *where $x(t,t_0,x_0)$ is a solution of the following system*

$$(3)\quad \frac{dx}{dt}=f(t,x),\ f(t,0)=0,\ \text{where } f(t,x)\in C(I\times R^n,R^n),$$ *and its perturbed system*

(4) $\frac{dx}{dt}=f(t,x)+g(t,x)$, *where* $g(t,x)\in C(I\times R^n,R^n)$,

then the zero solution of the system (3) is uniformly integrally stable.

In 1988, we proved the following theorem. (cf. [7].)

[Theorem 3.2] *Suppose that there exist functions* $V(t,x)\in C(I\times S_H,I)$ *and* $h(t)\in C(I,I)$, *which satisfies the follwing conditions:*

(i) $a(\|x\|)\leqq V(t,x)$, $V(t,0)=0$, *where* $a(t,r)$ *is continuous in* (t,r), *nondecreasing in* r *for each* t, *nondecreasing in* t *for each*, $a(t,r)>0$ *for any* $r\neq 0$ *and* $a(t,0)\equiv 0$,

(ii) $|V(t,x)-V(t,y)|\leqq M\|x-y\|$, $M>0$,

(iii) $\dot{V}_{(3)}(t,x)\leqq h(t)V(t,x(t,t_0,x_0))$ *with* $\int_0^\infty h(t)dt<\infty$, *where* $x(t,t_0,x_0)$ *is any solution of the system (3),*

then the zero solution of the system (3) is uniformly integrally stable.

For proof, see[7].

Before we state main result, we give the following theorems we shall need later on.

[Theorem 3.3] *The zero solution of the system (1) is partially uniformly integrally stable with respect to* x *if and only if for any* $\varepsilon>0$ *and any* $t_0\geqq 0$ *there exist* $\delta_1(\varepsilon)>0$ *and* $\delta_2(\varepsilon)>0$ *such that if* $\phi(t)$ *is continuous function defined on* $[t_0,\infty]$ *and satisfies* $\int_{t_0}^\infty\|\phi(t)\|dt<\infty$, *then any solution* $(u(t,t_0,u_0,v_0), v(t,t_0,u_0,v_0))$ *satisfying* $\|u_0\|+\|v_0\|<\delta_1(\varepsilon)$ *of the system*

$$(5)\quad \begin{cases} \dfrac{du}{dt}=f(t,u,v)+\phi(t) \\ \dfrac{dv}{dt}=g(t,u,v)+\phi(t) \end{cases}$$

verfies the inequality $\|u(t,t_0,u_0,v_0)\|<\varepsilon$ *for all* $t\geqq t_0$.

Proof. The necessity of the condition is clear. Therefore it suffices to prove that if the property from the statement occures, the zero solution of the system (1) is partially uniformly integrally stable with respect to x.

Let $F(t,u,v)$ and $G(t,u,v)$ be such that

$\int_{t_0}^\infty \sup_{\|u\|\leq\varepsilon}\{\|F(t,u,v)\|+\|G(t,u,v)\|\}dt<\delta_2(\varepsilon)$, where $\delta_2(\varepsilon)$ is one given in condition.

Consider the solution $(u(t,t_0,u_0,v_0), v(t,t_0,u_0,v_0))$ of the system (2) satisfying the condition $\|u_0\|+\|v_0\|<\delta_1(\varepsilon)$.

If we would not have for all $t\geqq t_0$ the inequality $\|u(t_0,u_0,v_0)\|<\varepsilon$, there exists a first point t_1 such that $\|u(t_1,t_0,u_0,v_0)\|=\varepsilon$ and $\|u(t,t_0,u_0,v_0)\|\leqq\varepsilon$ for all $t\in[t_0,t_1]$.

For any $t\in[t_0,t_1]$, take

$\phi(t)=F(t,u(t,t_0,u_0,v_0),v(t,t_0,u_0,v_0))$, we have

$$\int_{t_0}^{t_1}\|\phi(t)\|dt\leqq\int_{t_0}^{t_1}\sup_{\|u\|\leq\varepsilon}\{\|F(t,u,v)\|+\|G(t,u,v)\|\}dt<\delta_2(\varepsilon).$$

We extend $\phi(t)$ continuously on the whole semiaxis $t\geqq t_0$ such that $\int_{t_0}^\infty\|\phi^*(t)\|dt<\delta_2(\varepsilon)$, where $\phi^*(t)$ is a extended function, for this it is sufficient to take $t_2>t_1$ such that

$$t_2-t_1<\frac{2(\delta_2(\varepsilon)-\int_{t_0}^{t_1}\|\phi(t)\|dt)}{1+\|\phi(t_1)\|},$$

$\phi^*(t)=\phi(t)$ for all $t\in[t_0,t_1]$, linear on $[t_1,t_2]$, where we put $\phi^*(t_2)=0$, and zero for all $t\geqq t_2$.

We consider the following system

$$(6)\quad \begin{cases} \dfrac{du}{dt}=f(t,u,v)+\phi^*(t) \\ \dfrac{dv}{dt}=g(t,u,v)+\phi^*(t). \end{cases}$$

Now let be a solution $(u^*(t,t_o,u_o,v_o),v^*(t,t_o,u_o,v_o))$ of the system (6).

From $\|u_o\|+\|v_o\|<\delta_1(\varepsilon)$ and $\int_{t_o}^{\infty}\|\phi^*(t)\|\,dt<\delta_2(\varepsilon)$, we have $\|u^*(t,t_o,u_o,v_o)\|<\varepsilon$ for all $t\geqq t_o$: hence $\|u^*(t_1,t_o,u_o,v_o)\|<\varepsilon$. But we have $u^*(t,t_o,u_o,v_o)=u(t,t_o,u_o,v_o)$ on $[t_o,t_1]$, therefore $\|u(t_1,t_ou_o,v_o)\|<\varepsilon$, which is contradictory.

[Theorem 3.4] *The zero solution of the system (1) is partially uniformly integrally stable with respect to x if and only if for any* $\varepsilon>0$ *and any* $t_0\geqq 0$ *there exist* $\delta_1(\varepsilon)>0$ *and* $\delta_2(\varepsilon)>0$ *such that any* $t_1>t_0$ *whichever be* $p_1(t)$ *with continuous derivative on* $[t_0,t_1]$, *and* $p_2(t)$ *be continuous on* $[t_0,t_1]$ *satisfying* $\|p_1(t_0)\|+\|p_2(t_0)\|<\delta_1(\varepsilon)$, $\int_{t_0}^{\infty}\|p_1'(t)-f(t,p_1(t),p_2(t))\|\,dt<\delta_2(\varepsilon)$, $({}'=\frac{d}{dt})$, *it will follow that* $\|p_1(t)\|<\varepsilon$ *for all* $t\in[t_0,t_1]$.

Proof. Let us suppose that the zero solution of the system (1) is partially uniformly integrally stable with respect to x, let $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ be defined according to be the property of partially uniformly integral stability and let $p_1(t)$ and $p_2(t)$ be as in the statement.

Let $\phi^*(t)$ be continuous for all $t\geqq t_o$ such $\phi^*(t)=p'_1(t)-f(t,p_1(t),p_2(t))$ on $[t_o,t_1]$, linear on $[t_1,t_2]$, where t_2 is chosen such that

$$t_2-t_1<\frac{2(\delta_2(\varepsilon)-\int_{t_o}^{t_1}\|p_1'(t)-f(t,p_1(t),p_2(t))\|\,dt)}{1+\|p_1'(t_1)-f(t_1,p_1(t_1),p_2(t_1))\|},$$

and zero for all $t\geqq t_2$. Then $\int_{t_o}^{\infty}\|\phi^*(t)\|\,dt<\delta_2(\varepsilon)$.

We consider the system

$$(7)\quad \begin{cases} \dfrac{du}{dt}=f(t,u,v)+\phi^*(t) \\ \dfrac{dv}{dt}=g(t,u,v)+\phi^*(t), \end{cases}$$

and let $(u(t,t_o,p_1(t_o),p_2(t_o)),\ v(t,t_o,p_1(t_o),p_2(t_o))$ be the solution of the system (7).

According to Theorem 3.3, we get $\|u(t,t_o,p_1(t_o),\ p_2(t_o))\|<\varepsilon$ for all $t\geqq t_o$. Since we have $p_1(t)=u(t,t_o,p_1(t_o),p_2(t_o))$ on $[t_o,t_1]$, then it follow that $\|p_1(t)\|<\varepsilon$ for all $t\in[t_o,t_1]$.

Now let $\delta_1(\varepsilon)>0$ and $\delta_2(\varepsilon)>0$ be as in the statement of Theorem 3.4 and $\phi(t)$ be continuous on $[t_o,\infty]$ with $\int_{t_o}^{\infty}\|\phi(t)\|\,dt<\delta_2(\varepsilon)$.

For this function, we consider the system (5). Then the solution

$(u(t,t_o,p_1(t_o),p_2(t_o)),\ v(t,t_o,p_1(t_o),p_2(t_o)))$ such that $\|p_1(t_o)\|+\|p_2(t_o)\|<\delta_1(\varepsilon)$ verifies for any $t_1>t_o$ the condition of Theorem 3.4 : hence we get $\|u(t,t_o,p_1(t_o),p_2(t_o))\|<\varepsilon$ for all $t\in[t_o,t_1]$, it follow that $\|u(t,t_o,p_1(t_o),p_2(t_o))\|<\varepsilon$ for all $t\geqq t_o$.

Then, by Theorem 3.3, the zero solution of the system (1) is partially uniformly integrally stable with respect to x.

[Theorem 3.5] *Suppose that there exist functions* $V(t,x,y)\in C(I\times S_H\times R^m,I)$ *and* $h(t)\in C(I,I)$, *which satisfies the following conditions:*

(i) $a(t,\|x\|)\leqq V(t,x,y)$, $V(t,0,0)=0$, *where* $a(t,r)$ *is continuous in* (t,r), *nondecreasing in* r *for each* t, *nondecreasing in* t *for each* r, $a(t,r)>0$ *for all* $r\neq 0$ *and* $a(t,0)\equiv 0$, *(ii)* $|V(t,x_1,y_1)-V(t,x_2,y_2)| \leqq M\{\|x_1-x_2\|+\|y_1-y_2\|\}$, $M>0$,

(iii) $\dot{V}_{(1)}(t,x,y)\leqq h(t)V(t,x(t,t_0,x_0,y_0),\ y(t,t_0,x_0,y_0))$ *with* $h(t)\geqq 0$, $\int_0^\infty h(t)dt<\infty$,

where $(x(t,t_0,x_0,y_0),\ y(t,t_0,x_0,y_0))$ *is any solution of the system (1),*

then the zero solution of the system (1) is partially uniformly integrally stable with respect to x.

Proof. Let $\theta\in[t_0,t]$: for any function $p_1(t)$ with continuous derivatives on $[t_0,t]$ and any continuous function $p_2(t)$, we consider the solution $(x(t,\theta,p_1(\theta),p_2(\theta)),\ y(t,\theta,p_1(\theta),p_2(\theta)))$ of the system (1).

We have

$$|V(\theta+h,p_1(\theta+h),p_2(\theta+h)-V(\theta+h,x(\theta+h,\theta,p_1(\theta),p_2(\theta)),p_2(\theta+h))|$$
$$\leqq M\|p_1(\theta+h)-x(\theta+h,\theta,p_1(\theta),p_2(\theta))\|$$
$$=Mh\left\|\frac{1}{h}\int_\theta^{\theta+h}p_1'(t)dt-\frac{1}{h}\int_\theta^{\theta+h}f(t,x(t,\theta,p_1(\theta),p_2(\theta)),\ y(t,\theta,p_1(\theta),p_2(\theta)))dt\right\|,$$

hence

$$\overline{\lim_{h\to0+}}\frac{1}{h}[V(\theta+h,p_1(\theta+h),p_2(\theta+h))-V(\theta,p_1(\theta),p_2(\theta)]$$
$$\leqq\overline{\lim_{h\to0+}}\frac{1}{h}[V(\theta+h,p_1(\theta+h),p_2(\theta+h)-V(\theta+h,x(\theta+h,\theta,p_1(\theta),p_2(\theta)),p_2(\theta+h))]$$
$$+\overline{\lim_{h\to0+}}\frac{1}{h}[V(\theta+h,x(\theta+h,\theta,p_1(\theta),p_2(\theta)),p_2(\theta+h))-V(\theta,p_1(\theta),p_2(\theta))]$$
$$\leqq M\|p_1'(\theta)-f(\theta,p_1(\theta),p_2(\theta))\|+h(\theta)V(\theta,p_1(\theta),p_2(\theta)).$$

By integrating, we obtain

$$V(t,p_1(t),p_2(t))$$
$$\leqq V(t_0,p_1(t_0),\ p_2(t_0))\exp(\int_{t_0}^t h(u)du+M\int_{t_0}^t\exp(\int_{t_0}^s h(u)du)\|p_1'(s)-f(s,p_1(s),p_2(s))\|ds.$$

By the property of the function $h(t)$, there exists a positive constant $K>0$ such that $\int_0^\infty h(t)dt=K$.

For any $\varepsilon>0$ and any $t_0\geqq0$, by birtue of (i), we have constans $\delta_1(\varepsilon)>0$ and $\delta_2(\varepsilon)>0$ satisfying the following condition ; $Me^K(\delta_1(\varepsilon)+\delta_2(\varepsilon))<a(0,\varepsilon)$.

If $\|p_1(t_0)\|+\|p_2(t_0)\|<\delta_1(\varepsilon)$ and $\int_{t_0}^\infty\|p_1'(t)-f(t,p_1,p_2(t))\|dt<\delta_2(\varepsilon)$,

then $V(t,p_1(t),p_2(t))$

$$\leqq V(t_0,p_1(t_0),p_2(t_0))\exp(\int_{t_0}^t h(u)du)+M\int_{t_0}^t\exp(\int_{t_0}^s h(u)du)\|p_1'(s)-f(s,p_1(s),p_2(s))\|ds$$
$$\leqq Me^K(\|p_1(t_0)\|+\|p_2(t_0)\|)+Me^K\int_{t_0}^\infty\|p_1'(s)-f(s,p_1(s),p_2(s))\|ds$$
$$\leqq Me^K(\delta_1(\varepsilon)+\delta_2(\varepsilon)).$$

For any functions $p_1(t)$ and $p_2(t)$ satisfying $\|p_1(t_0)\|+\|p_2(t_0)\|<\delta_1(\varepsilon)$ and $\int_{t_0}^\infty\|p_1'(t)-f(t,p_1(t),p_2(t))\|dt<\delta_2(\varepsilon)$, we prove that $\|p_1(t)\|<\varepsilon$ for all $t\in[t_0,t_1]$.

If we assume that this is not true, then there exists t^* such that $\|p_1(t^*)\|\geqq\varepsilon$ and $t_0<t^*<t_1$.

By (i), it follows that

$$a(0,\varepsilon)\leqq a(t^*,\|p_1(t^*)\|)\leqq V(t^*,p_1(t^*),p_2(t^*))\leqq Me^K(\delta_1(\varepsilon)+\delta_2(\varepsilon))<a(0,\varepsilon),$$

which contradicts.

Therefore we have $\|p_1(t)\|<\varepsilon$ for all $t\in[t_0,t_1]$ for any functions $p_1(t)$ and $p_2(t)$ with $\|p_1(t_0)\|+\|p_2(t_0)\|<\delta_1(\varepsilon)$ and $\int_{t_0}^\infty\|p_1'(t)-f(t,p_1(t),p_2(t))\|dt<\delta_2(\varepsilon)$.

By Theorem 3.4, the zero solution of the system (1) is partially uniformly integrally stable with respect to x.

References

[1] A.Halanay : Differential Equations, Academic Press, 1966.

[2] A.Halanay : On the Asymptotic Behavior of the Solutions of an Integro-Differential Equations, J.Math.Anal.Appl., vol.10 (1965), pp.319 - 324.

[3] T.Yosizawa : Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions, Springer-Verlag, 1975.

[4] T.Yosizawa : Stability Theory by Liapunov's Second Method, The Math.Soci. of Japan, 1966.

[5] V.Lakshmikantham and S.Leela : Differential and Integral Inequalities, Vol.1, Ordinary Differential Equations, Academic Press, 1969.

[6] I.Vrkoc : Integral Stability, Czechoslovak Math.J., vol.9 (1959), pp.71 - 129.

[7] S.Seino : On the Uniformly Integral Stability of Solutions of Ordinary Differential Equations, Research Reports of Akita National College of Technology, vol.23 (1988), pp.60 - 66.

[8] M.Kudo : On the Integral Stability and the Uniformly Integral Stability of Nonlinear Differential Equations by Using Comparison Principle, Research Reports of Akita National College of Technology, vol.23 (1988), pp.67 - 72.

[9] M.Aso, M.Kudo and S.Seino : Asymptotically Integral Stability Theorems by the Comparison Principle, Research Reports of Akita National College of Technology, vol.23 (1988), pp.73 - 76.

[10] M.Kudo : Partially Integral Stability Theorems by the Comparison Principle, Research Reports of Akita National College of Technology, (to appear).

平成 2 年 2 月