On Partial L^p-Stability

Miki KUDO, Masamichi ASO and Shoichi SEINO

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1. Introduction

H. A. Antosiewicz [1] showed that if there exists a Liapunov function with a negative definite derivative for the system (E) $x' = f(t, x)$, then the zero solution of (E) is stable and every solution $x(t, t_0, x_0)$ starting with $\| x_0 \|$ sufficiently small tends to zero on some sequence of points $t_n \to \infty$.

What further restrictions are necessary and sufficient for every solution to be integrable on the half-line to the right of the initial point t_0 ?

To provide an answer to this question, A.Strauss [2] introduced a new kind of stability, i. e., L^P - stability.

Many authors have discussed the L^P- stability (cf. [3], [4] and [5]). We also have obtained some results of L^P - stability in the large of nonlinear differential - difference equations [6].

In many applications, we need to see the qualities not of the whole solution but of the partial. C.Corduneanu [7] and A.Halanay [8] have obtained some results of the partial stability. And we presented several results concerning it in [9], [10], [11] and [12].

In this paper, we describe several results concerning the partial L^P - stability of the solutions of differential equations.

C.Corduneanu [13] and H.A.Antosiewicz [14] observed that the Liapunov's second method depends basically on the fact that a function $u(t)$ satisfying the inequality $u' \leqq g(t,u(t))$ $(u(t_0) \leqq r_0)$ is majorized by the maximal solution of the scalar differential equation (E′) $r' = g(t,r), r(t_0) = r_0$. As the comparison principle reduces the problem of determining the behavior of the solution of (E) to the solution of a scalar equation (E′), it is a very impotant tool in applications. It is particularly useful in dealing with a variety of qualitative problem, see for example [15] We shall use this comparison technique for out results.

2. Notations and Definitions

Let R^+ denote the interval $[0,\infty)$ and R^k denote Euclidean k - space. For $x \in R^k$, let $\| x \|$ be any convenient norm, and we shall denote by S_ρ the set of x such that $\| x \| \leqq \rho$. As usual, we shall use ′ instead of d/dt.

Let E be an open (t,x) - set in R^{k+1}. We shall mean by $C[E, R^k]$ the class of continuous mappings from E into R^k. For brevity, K denotes the families of continuous increasing, positive definite functions.

Let us consider a differential system of the form

$$\begin{cases} x' = f_1(t, x, y), & x(t_0) = x_0, \\ y' = f_2(t, x. y), & y(t_0) = y_0, \end{cases} \tag{1}$$

where $f_1 \in C[J\times R^n \times R^m, R^n]$, $f_2 \in C[J\times R^n \times R^m, R^m]$ and J is a t-interval containing t_0. Let us denote a solution of (1) by $x(t) = x(t, t_0, x_0, y_0)$, $y(t)=y(t, t_0, x_0, y_0)$.

We define the function

$$D^+V(t, x, y)= \limsup_{h\to +0} \frac{1}{h} \{V(t+h, x+hf_1(t, x, y), y+hf_2(t, x, y)) - V(t, x, y)\}$$

for $(t, x, y) \in J\times R^n\times R^m$.

Further, we consider a scalar differential equation

$$u'=g(t, u), \qquad u(t_0)=u_0, \tag{2}$$

where $g \in C[J\times R^+, R]$ and $g(t, 0)= 0 \quad (\forall t \in J)$.

〔Definition 1〕 The zero solution of (1) is said to be partially L^p-stable with respect to x if it is partially stable with respect to x and there exists a $\delta_0 = \delta_0(t_0) > 0$ such that $\|x_0\| + \|y_0\| \leqq \delta_0$ implies

$$\int_{t_0}^{\infty} \|x(t, t_0, x_0, y_0)\|^p \, dt < \infty. \tag{3}$$

〔Definition 2〕 The zero solution of (1) is said to be partially uniformly L^p-stable with respect to x if it is partially uniformly stable with respect to x, the δ_0 in Definition 1 is independent of t_0 and the integral (3) converges uniformly in t_0.

〔Definition 3〕 The zero solution u=0 of (2) is said to be L^1-stable if it is stable and there exists a $\delta_1 = \delta_1(t_0) > 0$ such that $u_0 \leqq \delta_1$ implies

$$\int_{t_0}^{\infty} u(t, t_0, u_0)dt < \infty. \tag{4}$$

〔Definition 4〕 The zero solution u=0 of (2) is said to be uniformly L^1-stable if it is uniformly stable, the δ_1 in Definition 3 is independent of t_0 and the integral (4) converges uniformly in t_0

Preliminary Results

【Theorem 1】 *Let $V\in C[J\times R^n\times R^m, R^+]$ and $V(t,x,y)$ be locally Lipschitzian in x and y. Assume that the function $D^+V(t,x,y)$ satisfies*

$$D^+V(t,x,y)\leqq g(t, V(t,x,y)), \quad (t,x,y)\in J\times S\rho\times R^m,$$

where $g\in C[J\times R, R]$. Let $r(t)$ be the maximal solution of (2).

If $(x(t),y(t))$ is any solution of (1) such that $V(t_0,x_0,y_0)\leqq u_0$, then $V(t,x(t),y(t))\leqq r(t)$ for any $t\geqq t_0$.

For the proof, see [5].

【Theorem 2】 *Assume that there exist functions $V(t,x,y)$ and $g(t,u)$ satisfying the following conditions;*

(i) *$V\in C[J\times S\rho\times R^m, R^+]$, $V(t,0,0)=0$ $(\forall t\in J)$ and $V(t,x,y)$ is locally Lipschitzian in x and y, $g\in C[J\times R^+, R]$ and $g(t,0)=0$ $(\forall t\in J)$,*

(ii) *$a(t, \|x\|)\leqq V(t,x,y)$, where $a \in C[J\times R^+, R^+]$ is increases monotonically with respect to t for each fixed r, $a(t,0)=0$ $(\forall t\in J)$ and $a(t,r)>0$ for $r\neq 0$.*

(iii) *$D^+V(t,x,y) \leqq g(t, V(t,x,y))$, $(t,x,y)\in J\times S\rho\times R^m$.*

Then the stability of the zero solution of (2) implies the partial stability with respect to x of the zero solution of (1).

For the proof, see [11].

【Theorem 3】 *Assume that there exist functions $V(t,x,y)$ and $g(t,u)$ satisfying the following conditions;*

(i) *$V\in C[J\times S_\rho\times R^m, R^+]$, $V(t,0,0)=0$ $(\forall t\in J)$ and $V(t,x,y)$ is locally Lipschitzian in x* *and y, $g \in C[J\times R^+,R]$ and $g(t,0)=0$ ($\forall t\in J$),*

(ii) *$a(t,\|x\|) \leqq V(t,x,y) \leqq b(\|x\|+\|y\|)$, where $a\in C[J\times R^+,R^+]$ is increases monotonically with respect to t for each fixed r, $a(t,0)=0$ ($\forall t\in J$) and $a(t,r)>0$ for $r\neq 0$, and $b\in K$.*

(iii) *$D^+V(t,x,y) \leqq g(t,V(t,x,y))$, $(t,x,y)\in J\times S\rho\times R^m$.*

Then the uniformly stable of the zero solution of (2) implies partially uniformly stable with respect to x of the zero solution of (1).

For the proof, see [11].

【Theorem 4】 *Assume that there exists a function V(t,x,y) satisfying the following conditions;*

(i) *$V\in C[J\times S\rho\times R^m,R^+]$, $V(t,0,0)=0$ ($\forall t\in J$) and locally Lipschitzian in x and y,*

(ii) *$a(t,\|x\|)\leqq V(t,x,y)$, where $a\in C[J\times R^+,R^+]$ is increases monotonically with respect to t for each fixed r, $a(t,0)=0$ ($\forall t\in J$) and $a(t,r)>0$ for $r\neq 0$,*

(iii) *$D^+V(t,x,y)\leqq 0$.*

Then the zero solution of (1) is partially stable with respect to x.

For the proof, see [10].

【Theorem 5】 *Assume that there exists a function V(t,x,y) satisfying the following conditions;*

(i) *$V\in C[J\times S\rho\times R^m,R^+]$, $V(t,0,0)=0$ ($\forall t\in J$) and locally Lipschitzian in x and y,*

(ii) *$a(t,\|x\|)\leqq V(t,x,y)\leqq b(\|x\|+\|y\|)$, where $a\in C[J\times R^+,R^+]$ is increases monotonically with respect to t for each fixed r, $a(t,0)=0$ ($\forall t\in J$) and $a(t,r)>0$ for $r\neq 0$, and $b\in K$.*

(iii) *$D^+V(t,x,y)\leqq 0$.*

Then the zero solution of (1) is partially uniformly stable with respect to x.

<Proof> For any $\varepsilon>0$, we can choose a $\delta(\varepsilon)>0$ such that $\|x_0\|+\|y_0\|<\delta$ implies $b(\delta)<a(0,\varepsilon)$, because $b(r)\in K$.

Suppose that there exists some t_1 such that $\|x_0\|+\|y_0\|<\delta$ implies $\|x(t_1,t_0,x_0,y_0)\|=\varepsilon$, $t_1>t_0$. From the conditons (ii) and (iii), we have

$$\begin{aligned} a(0,\varepsilon) &\leqq a(t_1,\varepsilon)\\ &\leqq V(t_1,x(t_1),y(t_1))\\ &\leqq V(t_0,x(t_0),y(t_0))\\ &\leqq b(\|x_0\|+\|y_0\|)\\ &< b(\delta)\\ &< a(0,\varepsilon)\end{aligned}$$

This is contradiction. Therefore, the zero solution of (1) is partially uniformly stable with respect to x.

Main Results

【Theorem 6】 *Assume that there exist functions V(t,x,y) and g(t,u) satisfying the following conditions;*

(i) *$g\in C[J\times R^+,R]$ and $g(t,0)=0$ ($\forall t\in J$),*

(ii) *$V\in C[J\times S\rho\times R^m,R^+]$, $V(t,0,0)=0$ ($\forall t\in J$), V(t,x,y) is locally Lipschitzian in x and y, and $A\|x\|^p\leqq V(t,x,y)$, $A>0$, $(t,x,y)\in J\times S\rho\times R^m$,*

(iii) *$D^+V(t,x,y)\leqq g(t,V(t,x,y))$, $(t,x,y)\in J\times S\rho\times R^m$.*

Then the L^1-stability of the zero solution of (2) implies the partial L^p-stability with respect to x of the zero solution of (1).

<Proof> Since the zero solution u=0 of (2) is L^1- stable, it is stable and there exists a $\delta_1(t_0)$ such that $u_0\leqq\delta_1$ implies (4) . By Theorem 2, the zero solution of (1) is partially stable with respect to x.

We shall show that there exists a $\delta_0 = \delta_0(t_0)$ such that $\| x_0 \| + \| y_0 \| < \delta_0$ implies (3). Since V(t, x, y) is continuous and V(t, 0, 0)=0 ($\forall t \in J$), for given δ_1 and any $t_0 \in J$, there exists a positive number $\delta_0(t_0)$ satisfying the inequalities $\| x_0 \| + \| y_0 \| < \delta_0$, $V(t_0, x_0, y_0) \leqq \delta_1$ together. We set $u_0 = V(t_0, x_0, y_0)$. By using condition (ii) and Theorem 1, we have

$$V(t, x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0)) \leqq r(t, t_0, u_0), \quad t \geqq t_0,$$

where (x(t), y(t)) is any solution of (1) such that $\| x_0 \| + \| y_0 \| < \delta_0$ and $r(t, t_0, u_0)$ is maximal soluti on of (2). From this, it follows that

$$\begin{aligned} A \| x(t) \|^p &\leqq V(t, x(t), y(t)) \\ &\leqq r(t, t_0, u_0) \end{aligned}$$

and hence

$$\int_{t_0}^{\infty} \| x(t, t_0, x_0, y_0) \|^p dt \leqq 1/A \int_{t_0}^{\infty} r(t, t_0, u_0) dt < \infty.$$

Thus, the proof is complete.

【Theorem 7】 *Assume that there exist functions V(t,x,y) and g(t,u) satisfying the following conditions;*

(i) $g \in C[J \times R^+, R]$ *and* $g(t,0)=0$ ($\forall t \in J$),

(ii) $V \in C[J \times S_\rho \times R^m, R^+]$, $V(t,0,0)=0$ ($\forall t \in J$), *V(t,x,y) is locally Lipschitzian in x and y,* $A \| x \|^p \leqq V(t,x,y) \leqq b(\| x \| + \| y \|)$, $A>0$, $b \in K$,

(iii) $D^+ V(t,x,y) \leqq g(t, V(t,x,y))$, $(t,x,y) \in J \times S_\rho \times R^m$.

Then the uniform L^1-stability of the zero solution of (2) implies the partially uniform L^p-stability with respect to x of the zero solution of (1).

<Proof> From Theorem 3, the zero solution of (1) is partially uniformly stable with respect to x. Since the zero solution of (2) is uniformly L^1-stable, there exists a $\delta_1(\varepsilon)$ such that $u_0 \leqq \delta_1$ implies (4) and the integral (4) converges uniformly in t_0. To prove that the integral (3) converges uniformly in t_0, we follow the proof of Theorem 6 and choose $u_0 = b(\| x_0 \| + \| y_0 \|)$, thereby deducing $\delta_0 = b^{-1}(\delta_1)$. It is evident that δ_0 is independent of t_0 and the integral (3) coverges uniformly in t_0.

【Theorem 8】 *Assume that there exists a function V(t,x,y) satisfying the following conditions;*

(i) $V \in C[J \times S_\rho \times R^m, R^+]$, $V(t,0,0)=0$ ($\forall t \in J$) *and locally Lipschitzian in x and y,*

(ii) $a(t, \| x \|) \leqq V(t,x,y)$, *where* $a \in C[J \times R^+, R^+]$ *is increases monotonically with respect to t each fixed r,* $a(t,0)=0$ ($\forall t \in J$) *and* $a(t,r)>0$ *for* $r \neq 0$,

(iii) $D^+ V(t,x,y) \leqq -C \| x \|^p$, $(t,x,y) \in J \times S_\rho \times R^m$, $C>0$.

Then the zero solution of (1) is partially L^p-stable with respect to x.

<Proof> By Theorem 4, it follows that the zero solution of (1) is partilly stable with respect to x. Hence, there exists a $\delta_0 = \delta_0(t_0) > 0$ such that if $\| x_0 \| + \| y_0 \| \leqq \delta_0$, then $(t, x(t, t_0, x_0, y_0)) \in J \times S_\rho$ for $t \geqq t_0$. Define

$$m(t) = V(t, x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0)) + C\int_{t_0}^{t} \| x(t, t_0, x_0, y_0) \|^p dt.$$

By condition (iii), we have

$$m'(t) = D^+ V(t, x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0)) + C \| x(t, t_0, x_0, y_0) \|^p \leqq 0.$$

This implies that m(t) is nonincreasing, therefore

$$m(t) \leqq m(t_0)$$

$$\int_{t_0}^{\infty} \| x(t, t_0, x_0, y_0) \|^p dt \leqq m(t_0)/C = V(t_0, x_0, y_0)/C \quad \text{for } t \geqq t_0.$$

Thus the proof is completed.

【Theorem 9】 *Assume that there exists a function* $V(t,x,y)$ *satisfying the following conditions;*

(i) $V \in C[J \times S_\rho \times R^m, R^+]$, $V(t,0,0)=0\ (\forall t \in J)$ *and locally Lipschitzian in x and y,*

(ii) $a(t, \| x \|) \leqq V(t,x,y)$, *where* $a \in C[J \times R^+, R^+]$ *is increases monotonically with respect to t for each fixed r,* $a(t,0)=0\ (\forall t \in J)$ *and* $a(t,r)>0$ *for* $r \neq 0$

(iii) $D^+V(t,x,y) \leqq -C \| x \|^p$, $(t,x,y) \in J \times S_\rho \times R^m$, $C>0$.

Then the zero solution of (1) is partially uniformly L^p*-stable with respect to x.*

<Proof> Condition (iii), in virtue of (ii), reduces to $D^+V(t, x, y) \leqq g(t, V(t, x, y))$, where $g(t,u) = -(C/B)u$, and hence it is easy to check that the solution $u=0$ of (2) is uniformly L^1 - stable. Therefore, from Theorem 7, the zero solution of (1) is partially uniformly L^p - stable with respect to x.

References

〔1〕 H. A. Antosiewicz : A survey of Lyapunov's second method, Annals of Mathematics Studies No. 41 pp. 141 - 166, Princeton Univ. Press, Princeton, N. J, 1958.

〔2〕 A. Strauss : Liapunov functions and L^p - solutions of differential equations, Trans. Am. Math. Soc. 119, 1965, 37 - 50.

〔3〕 A. Strauss : On the stability of perturbed nonlinear systems, Proc. Amer. Math. Soc. 17, 1966, 803 - 807.

〔4〕 W. Hahn : On a new type of stability, J. Differential Eqs. 3, 1967, 440 - 448.

〔5〕 V. Lakshmikanthan and S. Leela : Differential and inequalities, Academic Press, 1969.

〔6〕 S. Seino, M. Kudo and M.Aso : L^p - stability in the large of nonlinear differential - difference equations, Reserch Reports of Akita Technical College, No. 10, 1975. 86 - 90.

〔7〕 C. Corduneanu : Sur la stabilite partielle, Rev. Math. Pures Appl. 9, 1964, 229 - 236.

〔8〕 A. Halanay : Asymptotic behavior of the solutions of some nonlinear integral equations, Rev. Math. Pures Appl. 10, 1965, 765 - 777.

〔9〕 M. Aso, M. Kudo and S. Seino : On partial stability of solutions of a system of ordinary differential equations, Reserch Reports of Akita Technical College, No. 16, 1981, 122 - 125.

〔10〕 S. Seino, M. Kudo and M. Aso : Partial stability by the comparison principle, Reserch Reports of Akita Technical College. No. 17, 1982, 95 - 99.

〔11〕 S. Seino, M. Kudo and M. Aso : Partial stability theorem by Liapunov's second method, Reserch Reports of Akita Technical College, No. 18, 1983, 132 - 135.

〔12〕 M. Kudo, M. Aso and S. Seino : On the partially total stability and the partially total boundedness of a system of ordinary differential equations, Reseach Reports of Akita National college of Technology, No. 20, 1985, 105 - 109.

〔13〕 C. Corduneanu : The applications of differential inequalities to stability theory, An. Sti. Univ. Al. I. Cuza, Iasi Sect. I a Mat., 6, 1960, 47 - 58.

〔14〕 H. A. Antosiewicz : An inequality for approximate solutions ordinary differential equations, Math. Z. 8, 1962, 44 - 52.

〔15〕 T. Yoshizawa : Stability theory by Liapunov's second method, Soc. Japan, 1966.