Asymptotically Integral Stability Theorems by the Comparison Principle

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1. Introduction

Liapunov's second method is widely recognized as an indispensable tool in studying not only the theory of stability but also many other qualitative properties of solutions of differential equations.

The main characteristic of this method is the introduction of the Liapunov function which defines a generalized distance from the origin of the phase space. As a result, the concept of Liapunov function furnishes a very general comparison principle which has been used in dealing with a variety of qualitative problems, that is, it reduces the problem of determining the behavior of solutions in a system of differential equations to the solution of a scalar differential equation.

In this paper, we present several results concerning the asymptotically integral stability bringing out the real significance of the comparison technique.

2. Definitions and Notations

Let R^n denote Euclidean n-space and R^+ denote the interval $[0, \infty)$. For $x \in R^n$, let $\|x\|$ be any convenient norm. We shall mean by $C(R^+ \times R^n, R^n)$ the class of continuous mappings from $R^+ \times R^n$ into R^n. For brevity, $\mathscr{K}$ denote the families of continuous increasing, positive definite functions.

We shall consider the differential system

$$x' = f(t, x) \tag{1}$$

and its perturbed system

$$x' = f(t, x) + R(t, x), \tag{2}$$

where $f, R \in C(R^+ \times R^n, R^n)$, $f(t, 0) \equiv 0$.

Furthermore, consider a scalar differential equation

$$u' = g(t, u) \tag{3}$$

and its perturbed system

$$u' = g(t, u) + \varphi(t), \tag{4}$$

where $g \in C(R^+ \times R^+, R)$, $\varphi \in C(R^+, R^+)$ and $g(t, 0) \equiv 0$.

For $V \in C(R^+ \times R^n, R^+)$, we define the function

$$D^+ V(t, x) = \limsup_{h \to +0} \frac{1}{h}\{V(t+h, x+hf(t, x)) - V(t, x)\}.$$

Occasionally, we write $D^+ V(t, x)_{(1)}$ to denote that the definition of $D^+ V(t, x)$ is with respect to the system (1).

Let $x(t, t_0, x_0)$ be any solution of (1) or (2) such that $x(t_0) = x_0$, and let $u(t, t_0, u_0)$ denote any solution of (3) or (4) such that $u(t_0) = u_0$.

Here, we give the definitions of integral stability.

[Definition 1] The zero solution $x = 0$ of (1) is said to be integrally stable, if for $\forall \alpha \geq 0$ and $\forall t_0 \in R^+$, there exists a positive function $\beta = \beta(t_0, \alpha)$, which is continuous in t_0 for each α and $\beta \in \mathscr{K}$ for each t_0 such that, for any solution $x(t, t_0, x_0)$ of the perturbed system (2), $\|x_0\| \leq \alpha$ and $\forall T > 0$,

$$\int_{t_0}^{t_0+T} \sup_{\|x\| \leq \beta} \|R(s, x)\| ds \leq \alpha$$

implies $\|x(t, t_0, x_0)\| < \beta$, $\forall t \geq t_0$.

[Definition 2] The zero solution $x = 0$ of (1) is said to be uniformly integrally stable, if the β in Definition 1 is independent of t_0.

[Definition 3] The zero solution $x = 0$ of (1) is said to be equi-integrally attractive, if for $\forall \varepsilon > 0$, $\forall \alpha \geq 0$ and $\forall t_0 \in R^+$, there exist positive numbers $T = T(t_0, \alpha, \varepsilon)$ and $\gamma = \gamma(t_0, \alpha, \varepsilon)$ such that, for any solutions of (2) and $\|x_0\| \leq \alpha$,

$$\int_{t_0}^{\infty} \sup_{\|x\| \leq \beta} \|R(s, x)\| ds < \gamma$$

implies $\|x(t, t_0, x_0)\| < \varepsilon$, $\forall t \geq t_0 + T$.

[Definition 4] The zero solution $x = 0$ of (1) is said to be equi-asymptotically integrally stabe, if it is integrally stable and is equi-asymptotically attractive.

[Definition 5] The zero solution $x = 0$ of (1) is said to be uniformly asymptotically integrally stable, if it is uniformly integrally stable and the T and the γ in Definition 3 are independent of t_0.

Corresponding to the definitions of integral stability (Definition 1-5), we designate by (Definition 6 -10) the concepts concerning the integral stability of the solution $u = 0$ in (3).

[Definition 6] The zero solution $u = 0$ of (3) is said to be integrally stable, if for $\forall \alpha_1 \geq 0$, $\forall t_0 \in R^+$, there exists a positive function $\beta_1 = \beta_1(t_0, \alpha_1)$, which is continuous in t_0 for each α_1 and $\beta_1 \in \mathscr{K}$ for each t_0 such that, for any solution $u(t, t_0, u_0)$ of the perturbed scalar equation (4), $u_0 \leq \alpha_1$ and $\forall T > 0$,

$$\int_{t_0}^{t_0+T} \varphi(s) ds \leq \alpha_1$$

implies $u(t, t_0, u_0) < \beta_1$, $\forall t \geq t_0$.

The definitions 7-10 may be formulated similarly.

3. Preliminary Results

[Theorem 1] *Suppose that the maximal solution $u(t)$ of (3) such that $u(t_0) = u_0$, stays on interval $[a, b]$. If a continuous function $x(t)$ with $x(t_0) \leq u_0$ satisfies*

$$x'(t) \leq g(t, x(t)),$$

where $g(t, u)$ is continuous on an open connected set $\Omega \subset R^2$, then we have $x(t) \leq u(t)$ for $a \leq t \leq b$.

For the proof, see references [1], [2] and [3].

[Theorem 2] *Assume that there exist functions $V(t, x)$ and $a(t, r)$ satisfying the following properties:*

(i) *$V \in C(R^+ \times R^n, R^+)$, $V(t, x)$ is Lipschitzian in x for a constant $M > 0$,*

(ii) *$a(t, \|x\|) \leq V(t, x)$, where $a \in C(R^+ \times R^+, R^+)$, $a(t, r) \in \mathscr{K}$ for each fixed t and monotone increasing with respect to t for each fixed r,*

(iii) *$D^+ V(t, x)_{(1)} \leq g(t, V(t, x))$.*

Then the integral stability of the zero solution $u = 0$ of (3) implies the integral stability of the zero solution $x = 0$ of (1).

For the proof, see reference [4].

[Theorem 3] *Under the assumptions of Theorem 2, if the zero solution $u = 0$ of (3) is uniformly integrally stable, then the zero solution $x = 0$ of (1) is uniformly integrally stable.*

For the proof, see reference [4].

[Theorem 4] *Assume that there exist function $V(t, x)$ and $a(r)$ satisfying the following properties :*

(i) $V \in C(R^+ \times R^n, R^+)$, *$V(t, x)$ is Lipschitzian in x for a constant $M > 0$,*

(ii) $a(\|x\|) \leq V(t, x)$, *where $a \in \mathcal{K}$ and $a(r) \to \infty$ uniformly in t as $r \to \infty$,*

(iii) $D^+ V(t, x)_{(1)} \leq g(t, V(t, x))$.

Then the equi-asymptotically integral stability of the zero solution $u = 0$ of (3) implies the equi-asymptotically integral stability of the zero solution $x(t) = 0$ of (1).

For the proof, see references [1] and [2].

[Theorem 5] *Under the assumptions of Theorem 4, if the zero solution $u = 0$ of (3) is uniformly asymptotically integrally stable, then the zero solution $x = 0$ of (1) is uniformly asymptotically integrally stable.*

For the proof, see references [1] and [2].

4. Main Results

[Theorem 6] *Assume that there exist function $V(t, x)$ and $a(t, r)$ satisfying the following properties :*

(i) $V \in C(R^+ \times R^n, R^+)$, *$V(t, x)$ is Lipschitzian in x for a constant $M > 0$,*

(ii) $a(t, \|x\|) \leq V(t, x)$, *where $a \in C(R^+ \times R^+, R^+)$, $a(t, r) \in \mathcal{K}$ for each fixed t, monotone increasing with respect to t for each fixed r and $a(t, r) \to \infty$ uniformly in t as $r \to \infty$,*

(iii) $D^+ V(t, x)_{(1)} \leq g(t, V(t, x))$.

Then the equi-asymptotically integral stability of the zero solution $u = 0$ of (3) implies the equi-asymptotically integral stability of the zero solution $x = 0$ of (1).

Proof. Since the zero solution $u = 0$ of (3) is equi-asymptotically integrally stable, the zero solution $u = 0$ of (3) is integrally stable. Hence, by Theorem 2, the zero solution $x = 0$ of (1) is integrally stable.

Now, we show that the zero solution $x = 0$ of (1) is equi-integrally attractive. For $\forall \varepsilon > 0$, $\forall \alpha \geq 0$ and $\forall t_0 \in R^+$, let $\|x_0\| \leq \alpha$ and define $\alpha_1 = M\alpha$. By the integral stability of the zero solution $x = 0$ of (1), there exists a $\beta = \beta(t_0, \alpha)$ such that

$$\int_{t_0}^{t_0+T} \sup_{\|x\| \leq \beta} \|R(s, x)\| ds \leq \alpha \text{ for } \forall T > 0$$

implies $\|x(t, t_0, x_0)\| < \beta(t_0, \alpha)$ for $\forall t \geq t_0$.

Since the zero solution of (3) is equi-integrally attractive, it follows that for given $a(t_0, \varepsilon) > 0$, $\alpha_1 \geq 0$, and $t_0 \in R^+$, there exists a pair of positive numbers $\gamma_1 = \gamma_1(t_0, \alpha_1, \varepsilon)$ and $T_1 = T_1(t_0, \alpha_1, \varepsilon)$ such that

$$\int_{t_0}^{\infty} \varphi(s) ds < \gamma_1, \; u_0 \leq \alpha_1 \tag{5}$$

implies

$$u(t, t_0, u_0) < a(t_0, \varepsilon) \text{ for } \forall t \geq t_0 + T_1, \tag{6}$$

where $\varphi \in C(R^+, R^+)$ and $u(t, t_0, u_0)$ is any solutions of (4).

Now, we set $\gamma = \gamma_1/M$ and $T = T_1$. Let $\{t_k\}$ be a sequence such that $t_k > t_0 + T$, $t_k \to \infty$ as $k \to \infty$. Suppose that there is a solution $x(t, t_0, x_0)$ of (2) such that $\|x_0\| \leq \alpha$ and $\|x(t_k, t_0, x_0)\| \geq \varepsilon$. From (i) and (iii), we have

$$D^+ V(t, x)_{(2)} \leq g(t, V(t, x(t))) + M \cdot \|R(t, x(t))\|. \tag{7}$$

If we define $\varphi(t) = M \cdot \|R(t, x(t))\|$, we have

$$\int_{t_0}^{\infty} \varphi(s)ds = \int_{t_0}^{\infty} M \cdot \|R(s, x(s))\| ds$$
$$\leq M \int_{t_0}^{\infty} \sup_{\|x\| \leq \beta} \|R(s, x(s))\| ds$$
$$< M \cdot \gamma = \gamma_1,$$

by using the fact that $\int_{t_0}^{\infty} \sup_{\|x\| \leq \beta} \|R(s, x(s))\| ds < \gamma$. This implies that, for solutions $u(t, t_0, u_0)$ of (4), (6) is true, because of (5). Moreover, by (7) and the definition of $\varphi(t)$, it follows from Theorem 1 that

$$V(t, x(t)) \leq r(t, t_0, u_0), \ ^{\forall} t \geq t_0, \tag{8}$$

where $r(t, t_0, u_0)$ is the maximal solution of (4). Hence, relations (6), (8) and the assumption (ii) lead us to the contradiction

$$a(t_0, \varepsilon) < a(t_k, \|x(t_k, t_0, x_0)\|)$$
$$\leq V(t_k, x(t_k, t_0, x_0))$$
$$\leq r(t_k, t_0, u_0)$$
$$< a(t_0, \varepsilon),$$

which proves the equi-integrally attractive of the zero solution $x = 0$ of (1). Therefore the zero solution $x = 0$ of (1) is equi-asymptotically integrally stable.

[Theorem 7] *Under the assumptions of Theorem* 5, *if the zero solution* $u = 0$ *of* (3) *is uniformly asymptotically integrally stable, then the zero solution* $x = 0$ *of* (1) *is uniformly asymptotically integrally stable.*

Proof. Since the zero solution $x = 0$ is uniformly integrally stable, there exists a $\beta = \beta(\alpha)$ such that for $\|x_0\| \leq \alpha$ and $^{\forall} T > 0$,

$$\int_{t_0}^{t_0+T} \sup_{\|x\| \leq \beta} \|R(s, x)\| ds \leq \alpha$$

implies $\|x(t, t_0, x_0)\| < \beta(\alpha)$ for $^{\forall} t \geq t_0$. By the uniformly integrally attractive, for given $a(0, \varepsilon) > 0$, $\alpha_1 = M\alpha$ and $t_0 \in R^+$, there exists a pair of positive numbers $\gamma_1 = \gamma_1(\alpha_1, \varepsilon)$ and $T_1 = T_1(\alpha_1, \varepsilon)$ such that

$$\int_{t_0}^{\infty} \varphi(s)ds < \gamma_1 \text{ for } u_0 \leq \alpha_1$$

implies $u(t, t_0, u_0) < a(0, \varepsilon)$ for $^{\forall} t \geq t_0 + T_1$.

Therefore, γ and T are independent of t_0. The rest of the proof is just the same with that of Theorem 5.

References

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