On the Integral Stability and the Uniformly Integral Stability of Nonlinear Differential Equations by Using Comparison Principle

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1. Introduction

Frequently, we cannot find the explicit solutions for the ordinary differential equations which is the object of study. But our purpose in solving most of the differential equations is to see their qualities rather than the concrete expressions of their solutions such as series or elementary functions.

A.M. Liapunov introduced a method which is called the second or direct method. [7] This method enables us to decide stability from the differential equations without any knowledge of their solutions.

Modifications and refinements of the concepts of stability have been introduced by many investigators. Some of them are complete stability, practical stability, strict stability, eventual stability, perfect stability, conditional stability and so on.

Dubošin introduced the concept of total stability (in the Soviet terminology, stable under constantly acting perturbations) which requires that the permanent perturbations be small all along. [8] In the case that we only require that the permanent perturbations be small in the mean, it is called to be stable under permanent perturbations bounded in the mean. [4]

Some type of stability, based on the idea of considering perturbations which can be great in certain moments but are small in the mean, has been defined by Ivo Vrkoč. It called integral stability. [1]

The equivalent notation of strong stability was defined by H. Okamura, and was stadied by K. Hayashi. [2]

Many authors have discussed the integral stability. [3], [4], [5]

The sufficient conditions in order that the properties of the integral stability and the uniformly integral stability of a scalar differential equation may inherit to those of a system of differential equations are shown in [4] and [5].

In this paper, we shall describe some improvements of these results.

2. Definitions and Notations

Let R^n denote Euclidean n-space and I denote the interval $[0, \infty)$. For $x \in R^n$, let $\|x\|$ be the Euclidean norm of x.

We shall denote by $C(I \times R^n, R^n)$ the set of all continuous functions defined on $I \times R^n$ with values in R^n.

A function $\phi(r)$ is said to belong to the class K if $\phi \in C(I, I)$, $\phi(0) = 0$, and $\phi(r)$ is strictly monotone increasing in r.

Let $' = d/dt$.

We shall study the system

$$x' = f(t, x), \qquad x(t_0) = x_0, \qquad t_0 \in I, \tag{1}$$

and its perturbed system

$$y' = f(t, y) + R(t, y), \qquad y(t_0) = y_0, \qquad t_0 \in I, \tag{2}$$

where $f, R \in C(I \times R^n, R^n)$, $f(t, 0) \equiv 0$.

For a continuous scalar function $V(t, x)$ defined on an open set, we define the function

$$D^+ V(t, x)_{(1)} = \limsup_{h \to +0} \frac{1}{h}[V(t+h, x+hf(t, x)) - V(t, x)].$$

In case $V(t, x)$ has continuous partial derivatives of first order, it is evident that

$$D^+ V(t, x)_{(1)} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t, x),$$

where "$\cdot$" denotes the scalar product.

Definition 1. ([5]) The trivial solution $x = 0$ of (1) is said to be equi-integrally stable, if for every $\alpha \geqslant 0$ and $t_0 \in I$, there exists a positive function $\beta = \beta(t_0, \alpha)$, which is continuous in t_0 for each α and $\beta \in K$ for each t_0, such that, for every solution $x(t, t_0, x_0)$ of the perturbed differential system (2), the inequality

$$\|x(t, t_0, x_0)\| < \beta, \; t \geqslant t_0,$$

hold, provided that $\|x_0\| \leqslant \alpha$ and, for every $T > 0$,

$$\int_{t_0}^{t_0+T} \sup_{\|x\| \leqslant \beta} \|R(s, x)\| ds \leqslant \alpha.$$

Definition 2. ([5]) The trivial solution $x = 0$ of (1) is said to be uniformly integrally stable if the β in Definition 1 is independent of t_0.

Definition 3. ([4]) The trivial solution $x = 0$ of (1) is said to be integrally stable, if there exists $\delta_0 > 0$ and a function $B(\delta) > 0$ defined for $0 < \delta < \delta_0$ (which we will suppose monotone and continuous) with $\lim_{\delta \to 0} B(\delta) = 0$ such that $\|y_0\| < \delta$ and

$$\int_{t_0}^{\infty} \sup_{\|y\| < B(\delta)} \|R(t, y)\| dt < \delta \text{ implies } \|y(t, y_0, y_0)\| < B(\delta)$$

for $t \geqslant t_0$, $y(t, t_0, y_0)$ being a solution of (2).

It is easy to see that this definition is equivalent to the following.

Definition 4. ([4]) The trivial solution $x = 0$ of (1) is said to be integrally stable, if for every $\varepsilon > 0$ and $t_0 \in I$, there exist positive functions $\delta_3 = \delta_3(t_0, \varepsilon)$ and $\delta_4 = \delta_4(t_0, \varepsilon)$ such that, for every solution $y(t, t_0, y_0)$ of the perturbed differential system (2), the inequality

$$\|y(t, t_0, y_0)\| < \varepsilon$$

holds for any $t \geqslant t_0$, provided that

$$\|y_0\| < \delta_3 \text{ and } \int_{t_0}^{\infty} \sup_{\|y\| \leqslant \varepsilon} \|R(t, y)\| dt < \delta_4.$$

Definition 5. The trivial solution $x = 0$ of (1) is said to be uniformly integrally stable, if the δ_3 and δ_4 in Definition 4 are independent of t_0.

We adopt Definition 4 and 5 as definitions of the integral stability and the uniformly integral stability respectively.

Further, we consider a scalar differential equation

$$u' = g(t, u), \qquad u(t_0) = u_0 > 0, \qquad t_0 \in I \tag{3}$$

and its perturbed differential equation

$$v' = g(t, v) + \phi(t), \qquad v(t_0) = v_0 \geqslant 0, \qquad t_0 \in I, \tag{4}$$

where $g \in C(I \times I, R)$, $g(t, 0) \equiv 0$.

Definition 6. ([5]) The trivial solution $u = 0$ of (3) is said to be equi-integrally stable if, for every $\alpha_1 > 0$, $t_0 \in I$, there exists a positive function $\beta_1 = \beta_1(t_0, \alpha_1)$ that is continuous in t_0 for each α_1 and $\beta_1 \in K$ for each t_0 such that, whichever be the function $\phi \in C(I, I)$ with

$$\int_{t_0}^{t_0+T} \phi(s)ds < \alpha_1$$

for every $T > 0$, every solution $v(t, t_0, v_0)$ of the perturbed scalar equation (4) satisfies the inequality

$$v(t, t_0, v_0) < \beta_1,\ t \geqslant t_0$$

provided $v_0 \leqslant \alpha_1$.

Definition 7. ([5]) The trivial solution $u = 0$ of (3) is said to be uniformly integrally stable if the function β_1 in Definition 7 is independent of t_0.

We define the integral stability and the uniformly integral stability of the trivial solution $u = 0$ of (3) as follows and adopt them for our results.

Definition 8. The trivial solution $u = 0$ of (3) is said to be integrally stable, if for every $\varepsilon > 0$ and $t_0 \in I$, there exist positive functions $\delta_1 = \delta_1(t_0, \varepsilon)$ and $\delta_2 = \delta_2(t_0, \varepsilon)$ such that, for every solution $v(t, t_0, v_0)$ of the perturbed differential equation (4), the inequality

$$v(t, t_0, v_0) < \varepsilon$$

holds for any $t \geqslant t_0$, provided that

$$v_0 < \delta_1 \text{ and } \int_{t_0}^{\infty} \phi(t)dt < \delta_2.$$

Definition 9. The trivial solution $u = 0$ of (3) is said to be uniformly integrally stable, if the functions δ_1 and δ_2 in Definition 8 are independent of t_0.

3. Preliminary Results

[Theorem 1] Assume that there exist functions $V(t, x)$ and $g(t, u)$ satisfying the following properties:

(i) $g \in C(I \times I, R)$, $g(t, 0) \equiv 0$.

(ii) $V \in C(I \times R^n, I)$, $V(t, x)$ is Lipschitzian in x for a constant $M > 0$, and there exists a function $b \in K$ such that $b(u) \to \infty$ as $u \to \infty$, and $b(\|x\|) \leqslant V(t, x)$, $(t, x) \in I \times R^n$.

(iii) $D^+V(t, x)_{(1)} \leqslant g(t, V(t, x))$, $(t, x) \in I \times R^n$.

Then, the equi-integral stability of the trivial solution $u = 0$ of (3) implies the equi-integral stability of the trivial solution $x = 0$ of (1).

For proof of this theorem, see [5].

[Theorem 2] Under the assumptions of Theorem 1, the uniformly integral stability of the trivial solution $u = 0$ of (3) assures the uniformly integral stability of the trivial solution $x = 0$ of (1).

For proof of this theorem, see [5].

[Theorem 3] The trivial solution of system (1) is integrally stable if and only if there exists $\delta_0 > 0$ and $B(\delta)$ defined on $(0, \delta_0)$, $B(\delta) > 0$, $\lim_{\delta \to 0} B(\delta) = 0$, such that whichever be the function $\phi(t)$ continuous, with $\int_{t_0}^{\infty} \phi(t)dt < \delta$, the solution $v(t, t_0, v_0)$ of the system (4) with $v_0 < \delta$ verifies the inequality $v(t, t_0, v_0) < B(\delta)$ for $t \geqslant t_0$.

For proof of this theorem, see [4].

In proofs of these theorems, the following comparison theorem plays an important role. And for obtaining our results, we will use it too.

[Theorem 4] Let E be an open (t, u)-set in R^2 and $h \in C(E, R)$. Suppose that $[t_0, t_0+a)$ is the largest interval in which the maximal solution $r(t)$ of

$$u' = h(t, u),\ u(t_0) = u_0$$

exists. Let $m \in C((t_0, t_0+a), R)$, $(t, m(t)) \in E$ for $t \in [t_0, t_0+a)$, $m(t_0) \leqslant u_0$, and for a Deni derivative D,

$$Dm(t) \leqslant h(t, u),\ t \in [t_0, t_0+a) - S,$$

where S is an atmost countable subset of $[t_0, t_0+a)$. Then,

$$m(t) \leqslant r(t),\ t \in [t_0, t_0+a).$$

For proof of this theorem, see [5].

4. Main Results

[Theorem 5]

Assume that there exists functions $V(t, x)$ and $a(t, r)$ satisfying the following properties :

(i) $V \in C(I \times R^n, I)$, *$V(t, x)$ is Lipschitzian in x for a constant $M > 0$,*

(ii) *$a(t, \|x\|) \leqslant V(t, x)$, where $a \in C(I \times I, I)$ and monotone increasing with respect to t for each fixed r and $a(t, r) > 0$ for $r \neq 0$,*

(iii) *$D^+V(t, x)_{(1)} < g(t, V(t, x))$, $(t, x) \in I \times R^n$.*

Then, the integral stability of the trivial solution $u = 0$ of (3) *implies the integral stability of the trivial solution $x = 0$ of* (1).

(Proof) For any given $\varepsilon > 0$ and $t_0 \in I$, a positive number $\varepsilon_1 = a(t_0, \varepsilon)$ is decided. For this ε_1, by the integral stability of the trivial solution $u = 0$ of (3), there exist positive numbers $\delta_1 = \delta_1(t_0, \varepsilon_1) = \delta_1(t_0, a(t_0, \varepsilon))$ and $\delta_2 = \delta_2(t_0, \varepsilon_1) = \delta_2(t_0, a(t_0, \varepsilon))$ such that, for the solution $v(t, t_0, v_0)$ of (4), the inequality $v(t, t_0, v_0) < \varepsilon_1$ holds for any $t \geqslant t_0$, provided that

$$v_0 < \delta_1 \text{ and } \int_{t_0}^{\infty} \phi(t)dt < \delta_2.$$

By the continuity of the function $V(t, x)$, there exists a positive number $\delta_3 = \delta_3(t_0, \varepsilon)$ such that

$$\|y_0\| < \delta_3 \text{ implies } V(t_0, y_0) < \delta_1. \tag{5}$$

Let $y(t) = y(t, t_0, y_0)$ be any solution of (2) such that $\|y_0\| < \delta_3$.

Since $V(t, y)$ is Lipschitzian in y for a constant $M > 0$, by (iii), we have

$$D^+v(t, y)_{(2)} \leqslant g(t, V(t, y)) + M\|R(t, y)\|$$

as far as $y(t)$ exists to the right of t_0.

Define $\lambda(t) = M\|R(t, y(t))\|$, and choose $w_0 = V(t_0, y_0)$.

By (5), an application of Theorem 4 shows that

$$V(t, y(t)) \leqslant r(t, t_0, w_0) \tag{6}$$

on the common interval of existence of $y(t)$ and $r(t, t_0, w_0)$, where $r(t, t_0, w_0)$ is the maximal solution of

$$w' = g(t, w) + \lambda(t),\ w(t_0) = w_0.$$

Let $\delta_4 = \delta_4(t_0, \varepsilon) = \delta_2(t_0, a(t_0, \varepsilon))/M$. Assume that there exists a t_1 such that

$$\|y(t_1, t_0, y_0)\| = \varepsilon \tag{7}$$

and $\quad \|y(t, t_0, y_0)\| \leqslant \varepsilon$ for $t \in [t_0, t_1]$

as $\quad \int_{t_0}^{\infty} \sup_{\|y\| \leqslant \varepsilon} \|R(t, y(t))\| dt < \delta_4.$

For $t \in [t_0, t_1]$, take $\phi(t) = M\|R(t, y(t))\|$.

Then we have

$$\int_{t_0}^{t_1} \phi(t) dt = \int_{t_0}^{t_1} M\|R(t, y(t))\| dt$$

$$\leqslant M \int_{t_0}^{t_1} \sup_{\|y\| \leqslant \varepsilon} \|R(t, y(t))\| dt < M\delta_4 = \delta_2.$$

We can extend $\phi(t)$ continuously for all $t \geqslant t_0$ such that

$$\int_{t_0}^{\infty} \phi(t) dt < \delta_2. \tag{8}$$

Let $p(t, t_0, w_0)$ be the maximal solution of the perturbed differential equation

$$v' = g(t, v) + \phi(t), \quad v(t_0) = w_0,$$

with $\phi(t)$ chosen as before. On $[t_0, t_1]$, we have $p(t, t_0, w_0) = r(t, t_0, w_0)$ since $\lambda(t)$ and $\phi(t)$ are identical on this interval. Particularly

$$p(t_1, t_0, w_0) = r(t_1, t_0, w_0). \tag{9}$$

By the integral stability of the trivial solution $u = 0$ of (3), from relations (5) and (8), we have

$$p(t, t_0, w_0) < \varepsilon_1 \text{ for any } t \geqslant t_0. \tag{10}$$

Thus, we get, from relations (7), (6), (9), (10) and assumption (ii), the following contradiction:

$$a(t_0, \varepsilon) = a(t_0, \|y(t_1, t_0, y_0)\|) < a(t_1, \|y(t_1, t_0, y_0)\|)$$

$$\leqslant V(t_1, y(t_1, t_0, y_0)) \leqslant r(t_1, t_0, w_0) = p(t_1, t_0, w_0) < \varepsilon_1.$$

Therefore we have $\|y(t, t_0, y_0)\| < \varepsilon$ for any $t \geqslant t_0$.

Thus, we have that the trivial solution $x = 0$ of (1) is integrally stable.

[Theorem 6]

Under the assumptions of Theorem 5, the uniformly integral stability of the trivial solution $u = 0$ *of* (3) *assures the uniformly integral stability of the trivial solution* $x = 0$ *of* (1).

(Proof) For any given $\varepsilon > 0$, a positive number $a(0, \varepsilon)$ is decided. For this $a(0, \varepsilon)$, by the uniformly integral stability of the trivial solution $u = 0$ of (3), there exist positive numbers $\delta_1 = \delta_1(\varepsilon)$ and $\delta_2 = \delta_2(\varepsilon)$ such that, for the solution $v(t, t_0, v_0)$ of (4), the inequality $v(t, t_0, v_0) < a(0, \varepsilon)$ holds for any $t \geqslant t_0$, provided that

$$v_0 < \delta_1 \text{ and } \int_{t_0}^{\infty} \phi(t) dt < \delta_2.$$

Let $\delta_3(\varepsilon) = \delta_1(\varepsilon)/M$ and $\delta_4(\varepsilon) = \delta_2(\varepsilon)/M$ where M is a Lipschitz constant.

If we assume that there exists a t_1 such that

$$\|y(t_1, t_0, y_0)\| = \varepsilon \text{ and}$$

$$\|y(t, t_0, y_0)\| \leqslant \varepsilon \text{ for } t \in [t_0, t_1] \text{ as } \|y_0\| < \delta_3$$

and $\int_{t_0}^{\infty} \sup_{\|y\| \leqslant \varepsilon} \|R(t, y(t))\| dt < \delta_4$, where $y(t) = y(t, t_0, y_0)$ is any solution of (2), then the same computations as in Theorem 5 lead to the following contradiction:

$$a(0, \varepsilon) < a(t_1, \varepsilon) = a(t_1, \|y(t_1, t_0, y_0)\|)$$
$$\leqslant V(t_1, \|y(t_1, t_0, y_0)\|) \leqslant r(t_1, t_0, w_0) = p(t_1, t_0, w_0) < a(0, \varepsilon).$$

Therefore the trivial solution $x = 0$ of (1) is uniformly integrally stable.

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