On the Uniformly Integral Stability of Solutions of Ordinary Differential Equations

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(Received on 19, October, 1987)

1. Introduction

A variant of the notion of total stability is obtained if instead of requiring that the permanent perturbations be small all along, we only require that they be small in the mean. A slightly different variant of the same type of stability, equally based on the idea of considering perturbations which can be great in certain moments but are small in the mean, has been defined by Ivo Vrkoc. It is called integral stability.

Many authers have discussed the integral stabilty. (cf. [1], [2], [3], [5], [6].)

As is well known, Liapunov's second method has its origin in three simple theorems that form the core of what he himself called his second method for dealing with questions of stability. It is widely recognized, today, as an indispensable tool not only in the theory of stability but also in studying many other qualitative properties of solutions of differential equations. The main characteristic of this method is the introduction of a function, namely the Liapunov function, which defines a generalized distance from the origin of the motion space.

Liapunov's second method is a very useful and powerful instrument in discussing the stability of the system of differential equations. Its power and usefulness lie in the fact that the decision is made by investigating the differential equation itself and not by finding solutions of the differential equations. However, it is great difficult to find the Liapunov function satisfying certain conditions. Therefore, it is important to obtain a weak sufficient condition for a stability theorem.

In this paper, by using the Liapunov's second method, we will state some extension of the sufficient conditions for the uniformly integral stability and the uniformly asymptotically integral stability.

2. Notations and Definitions

First, we summarize some basic notations and definition we will need later on.

Let I denote the interval $0 \leqq t < \infty$, R^n denote Euclidean n-space and R^+ denote the nonnegative real line. For $x \in R^n$, let $\|x\|$ be any norm of x and we shall denote by S_H the set of x such that $\|x\| < H$, $H > 0$.

We consider a system of differential equations

$$\frac{dx}{dt} = f(t, x), \tag{1}$$

where x is an n-vector and $f(t, x)$ is an n-vector functions.

Suppose that $f(t, x)$ is continuous on $I \times R^n$ and that $f(t, x)$ is smooth enough to ensure existence, uniqueness and continuous dependence of the solutions of the initial value problem associated with (1), moreover, $f(t, 0) \equiv 0$.

We shall denote by $C(I \times R^n, R^n)$ the set of all continuous functions defined on $I \times R^n$ with valued in R^n. Throughout this paper, a solution through a point $(t_0, x_0) \in I \times R^n$ will be denoted by such a form as $x(t, t_0, x_0)$.

We introduce the following definitions.

〔Definition 2.1〕 Corresponding to a continuous scalar function $V(t, x)$ defined on an open set, we define the function

$$\dot{V}_{(1)}(t, x) = \overline{\lim_{h \to 0+}} \frac{1}{h}\{V(t+h, x+hf(t, x)) - V(t, x)\}.$$

In case $V(t, x)$ has continuous partial derivatives of the first order, it is evident that

$$\dot{V}_{(1)}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t, x),$$

where "·" denote a scalar product.

3. Uniformly Integral Stability

We consider the differential equations

(1) $$\frac{dx}{dt} = f(t, x), \ f(t, 0) \equiv 0, \text{ where } f(t, x) \in C(I \times R^n, R^n),$$

and its perturbed system

(2) $$\frac{dx}{dt} = f(t, x) + g(t, x), \text{ where } g(t, x) \in C(I \times R^n, R^n).$$

Let us first define the notion of the integral stability.

〔Definition 3.1〕 The zero solution of the system (1) is said to be integrally stable if for any $\varepsilon < 0$ and any $t_0 \geqq 0$ there exist $\delta_1(t_0, \varepsilon) > 0$ and $\delta_2(t_0, \varepsilon) > 0$ such that $\|x_0\| < \delta_1(t_0, \varepsilon)$ and

$$\int_{t_0}^{\infty} \sup_{\|x\| \leqq \varepsilon} \|g(t, x)\| dt < \delta_2(t_0, \varepsilon) \text{ implies } \|x(t, t_0, x_0)\| < \varepsilon \text{ for all } t \geqq t_0,$$

where $x(t, t_0, x_0)$ denote a solution of the system (2) satisfying a initial condition $x(t_0, t_0, x_0) = x_0$.

〔Definition 3.2〕 The zero solution of the system (1) is said to be uniformly integrally stable if δ_1 and δ_2 in Definition 3.1 are independent of t_0.

It is easy to see that Definition 3.2 is equivalent to the following.

〔Definition 3.3〕 The zero solution of the system (1) is said to be uniformly integrally stable if for any $\alpha > 0$ and $t_0 \geqq 0$ there exists $\beta = \beta(\alpha) > 0$ such that the inequality $\|x_0\| \leqq \alpha$ and

$$\int_{t_0}^{\infty} \sup_{\|x\| \leqq \beta} \|g(t, x)\| dt < \alpha \text{ implies } \|x(t, t_0, x_0)\| < \beta(\alpha) \text{ for all } t \geqq t_0.$$

For proof, see 〔3〕.

In 〔3〕, A. Halanay proved the next theorem.

〔Theorem 3.1〕 *If there exists a continous function $V(t, x)$ defined on $I \times S_H$ with properties :*

(i) $a(\|x\|) \leqq V(t, x)$, $V(t, 0) \equiv 0$, *where $a(r)$ is a continuous, positive definite and increasing function,*

(ii) $|V(t, x) - V(t, y)| \leqq M\|x - y\|$, $M > 0$,

(iii) $\dot{V}_{(1)}(t, x) \leqq g(t) V(t, x(t, t_0, x_0))$, *with* $\int_0^{\infty} g(t) dt < \infty$, $g(t) \geqq 0$, *where $x(t, t_0, x_0)$ is a solution of the system (1),*

then the zero solution of the system (1) is uniformly integrally stable.

Before we state main result, we give the following lemmas we shall need later on.

〔Lemma 3.2〕 *The zero solution of the system (1) is uniformly integrally stable if and only if for any $\varepsilon > 0$ and $t_0 \geqq 0$ there exist $\delta_1(\varepsilon) > 0$ and $\delta_2(\varepsilon) > 0$ such that if $\varphi(t)$ is any continuous function defined on $[t_0, \infty)$ and satisfies*

$$\int_{t_0}^{\infty} \|\varphi(t)\| dt < \delta_2(\varepsilon),$$

then any solution $y(t, t_0, y_0)$ *satisfying* $\|y_0\| < \delta_1(\varepsilon)$ *of the system*

(3) $$\frac{dy}{dt} = f(t, y) + \varphi(t)$$

verifies the inequality $\|y(t, t_0, y_0)\| < \varepsilon$ *for all* $t \geqq t_0$.

Proof. The necessity of the condition is clear. Therefore it suffices to prove that if the property from the statement occures, the zero solution of the system (1) is uniformly integrally stable.

Let $g(t, x)$ be such that $\int_{t_0}^{\infty} \sup_{\|x\| \leqq \varepsilon} \|g(t, x)\| dt < \delta_2(\varepsilon)$, where $\delta_2(\varepsilon)$ is one given in the condition.

Consider a point x_0 with $\|x_0\| < \delta_1(\varepsilon)$ and the solution $x(t, t_0, x_0)$ of the system (2).

If we would not have for all $t \geqq t_0$ the inequality $\|x(t, t_0, x_0)\| < \varepsilon$, there exists a first point t_1 such that $\|x(t_1, t_0, x_0)\| = \varepsilon$ and $\|x(t, t_0, x_0)\| \leqq \varepsilon$ for all $t \in [t_0, t_1]$. For any $t \in [t_0, t_1]$ take $\varphi(t) = g(t, x(t, t_0, x_0))$, we have

$$\int_{t_0}^{t_1} \|\varphi(t)\| dt \leqq \int_{t_0}^{t_1} \sup_{\|x\| \leqq \varepsilon} \|g(t, x)\| dt < \delta_2(\varepsilon).$$

We extend $\varphi(t)$ continuously on the whole semiaxis $t \geqq t_0$ such that $\int_{t_0}^{\infty} \|\varphi^*(t)\| dt < \delta_2(\varepsilon)$, where $\varphi^*(t)$ is a extended function, for this it is sufficient to take $t_2 > t_1$ such that

$$t_2 - t_1 < \frac{2\left(\delta_2(\varepsilon) - \int_{t_0}^{t_1} \|\varphi(t)\| dt\right)}{1 + \|\varphi(t_1)\|},$$

$\varphi^*(t) = \varphi(t)$ for all $t \in [t_0, t_1]$, linear on $[t_1, t_2]$, where we put $\varphi^*(t_2) = 0$, and zero for all $t \geqq t_2$.

We consider the following system (4).

(4) $$\frac{dy}{dt} = f(t, y) + \varphi^*(t).$$

Now let $y(t, t_0, x_0)$ be a solution of the system (4).

From $\|x_0\| < \delta_1(\varepsilon)$ and $\int_{t_0}^{\infty} \|\varphi^*(t)\| dt < \delta_2(\varepsilon)$, we have $\|y(t, t_0, x_0)\| < \varepsilon$ for all $t \geqq t_0$: hence $\|y(t_1, t_0, x_0)\| < \varepsilon$. But we have $y(t, t_0, x_0) = x(t, t_0, x_0)$ on $[t_0, t_1]$, hence $\|x(t_1, t_0, x_0)\| < \varepsilon$, which is contradictory.

[Lemma 3.3] *The zero solution of the system (1) is uniformly integrally stable if and only if for any* $\varepsilon > 0$ *and* $t_0 \geqq 0$ *there exist* $\delta_1(\varepsilon) > 0$ *and* $\delta_2(\varepsilon) > 0$ *such that for any* $t_1 > t_0$ *whichever be* $y(t)$ *with continuous derivative on* $[t_0, t_1]$ *with* $\|y(t_0)\| < \delta_1(\varepsilon)$, $\int_{t_0}^{t_1} \left\| \frac{dy}{dt} - f(t, y(t)) \right\| dt < \delta_2(\varepsilon)$, *it will follow that* $\|y(t)\| < \varepsilon$ *for all* $t \in [t_0, t_1]$.

Proof. Let us suppose that the zero solution of (1) is uniformly integrally stable, let $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ be defined according to be the property of uniformly integral stability and let $y(t)$ be as in the statement. Let $\varphi^*(t)$ be continuous for all $t \geqq t_0$ such that
$\varphi^*(t) = y(t) - f(t, y(t))$ on $[t_0, t_1]$, linear on $[t_1, t_2]$, where t_2 is chosen such that

$$t_2 - t_1 < \frac{2\left(\delta_2(\varepsilon) - \int_{t_0}^{t_1} \left\| \frac{dy}{dt} - f(t, y(t)) \right\| dt\right)}{1 + \left\| \left(\frac{dy}{dt}\right)_{t=t_1} - f(t_1, y(t_1)) \right\|},$$

and zero for $t \geqq t_2$. Then $\int_{t_0}^{\infty} \|\varphi^*(t)\| dt < \delta_2(\varepsilon)$.

We consider the system

(5) $$\frac{dz}{dt} = f(t, z) + \varphi^*(t),$$

and let $z(t, t_0, y(t_0))$ be the solution of the system (5).

According to Lemma 3.2, we get $\|z(t, t_0, y(t_0))\| < \varepsilon$ for all $t \geqq t_0$.

Since we have $y(t) = z(t, t_0, y(t_0))$ on $[t_0, t_1]$, then it follow that $\|y(t)\| < \varepsilon$ for all $t \in [t_0, t_1]$.

Now let $\delta_1(\varepsilon) > 0$ and $\delta_2(\varepsilon) > 0$ be as in the statement of Lemma 3.3 and $\varphi(t)$ be continuous on $[t_0, \infty)$ with $\int_{t_0}^{\infty} \|\varphi(t)\| dt < \delta_2(\varepsilon)$. For this function, we consider the system (3). Then the solution $y(t, t_0, y_0)$ such that $\|y_0\| < \delta_1(\varepsilon)$ of the system (3) verifies for any $t_1 > t_0$ the condition of Lemma 3.3: hence we get $\|y(t, t_0, y_0)\| < \varepsilon$ for all $t \in [t_0, t_1]$, it follow that $\|y(t, t_0, y_0)\| < \varepsilon$ for all $t \geqq t_0$.

Then, by Lemma 3.2, the zero solution of (1) is uniformly integrally stable.

〔Theorem 3.4〕 *Suppose that there exist functions* $V(t, x) \in C(I \times S_H, R^+)$ *and* $g(t) \in C(I, R^+)$, *which satisfies the following conditions:*

(i) $a(t, \|x\|) \leqq V(t, x)$, $V(t, 0) \equiv 0$, *where* $a(t, r)$ *is continuous in* (t, r), *nondecreasing in* r *for each* t, *nondecreasing in* t *for each* r, $a(t, r) > 0$ *for any* $r \neq 0$ *and* $a(t, 0) \equiv 0$,

(ii) $|V(t, x) - V(t, y)| \leqq M\|x - y\|$, $M > 0$,

(iii) $\dot{V}_{(1)}(t, x) \leqq g(t) V(t, x(t, t_0, x_0))$ *with* $\int_0^{\infty} g(t) dt < \infty$, $g(t) \geqq 0$, *where* $x(t, t_0, x_0)$ *is any solution of the system (1),*

then the zero solution of the system (1) is uniformly integrally stable.

Proof. Let $\theta \in [t_0, t]$; for any function $y(t)$ with continuous derivative on $[t_0, t]$, we consider the solution $x(t, \theta, y(\theta))$ of the system (1).

We have

$$|V(\theta+h, y(\theta+h)) - V(\theta+h, x(\theta+h, \theta, y(\theta)))| \leqq M\|y(\theta+h) - x(\theta+h, \theta, y(\theta))\|$$

$$= Mh\left\| \int_{\theta}^{\theta+h} \frac{dy(t)}{dt} dt - \frac{1}{h} \int_{\theta}^{\theta+h} f(t, x(t, \theta, y(\theta))) dt \right\|,$$

hence,

$$\overline{\lim_{h \to 0+}} \frac{1}{h} [V(\theta+h, y(\theta+h)) - V(\theta+h, x(\theta+h, \theta, y(\theta)))]$$

$$\leqq M\left\| \left(\frac{dy}{dt}\right)_{t=\theta} - f(\theta, y(\theta)) \right\| + g(\theta) V(\theta, y(\theta)).$$

By integrating, we obtain

$$V(t, y(t)) \leqq V(t_0, y(t_0)) \exp\left(\int_{t_0}^{t} g(u) du\right) + M \int_{t_0}^{t} \exp\left(\int_{\theta}^{t} g(u) du\right) \left\| \frac{dy(s)}{ds} - f(s, y(s)) \right\| ds.$$

By the property of the function $g(t)$, there exists a positive constant $K > 0$ such that $\int_{t_0}^{\infty} g(u) du = K$.

For any $\varepsilon > 0$ and any $t_0 \geqq 0$, by birtue of (i), we have constants $\delta_1(\varepsilon) > 0$ and $\delta_2(\varepsilon) > 0$ satisfying the following condition:

$$Me^K (\delta_1(\varepsilon) + \delta_2(\varepsilon)) < a(0, \varepsilon).$$

If $\|y(t_0)\| < \delta_1(\varepsilon)$ and $\int_{t_0}^{\infty} \left\| \frac{dy(t)}{dt} - f(t, y(t)) \right\| dt < \delta_2(\varepsilon)$,

then

$$V(t, y(t)) \leqq M\|y(t_0)\| e^K + M \int_{t_0}^{t} e^K \left\| \frac{dy(s)}{ds} - f(s, y(s)) \right\| ds$$

$$\leqq Me^K \delta_1(\varepsilon) + Me^K \delta_2(\varepsilon).$$

For any function $y(t)$ satisfying $\|y(t_0)\| < \delta_1(\varepsilon)$ and $\int_{t_0}^{\infty} \left\| \frac{dy(t)}{dt} - f(t, y(t)) \right\| dt < \delta_2(\varepsilon)$, we prove that

$\|y(t)\| < \varepsilon$ for all $t \in [t_0, t_1]$.

If we assume that this is not true, then there exists t^* such that $\|y(t^*)\| \geqq \varepsilon$ and $t_0 < t^* < t_1$.

By (i), it follows that

$$a(0, \varepsilon) \leqq a(t^*, y(t^*)) \leqq V(t^*, y(t^*)) \leqq Me^K(\delta_1(\varepsilon)+\delta_2(\varepsilon)),$$

which contradicts.

Therefore we have $\|y(t)\| < \varepsilon$ for all $t \in [t_0, t_1]$ for any function $y(t)$ with $\|y(t_0)\| < \delta_1(\varepsilon)$ and $\int_{t_0}^{\infty} \left\| \frac{dy(t)}{dt} - f(t, y(t)) \right\| dt < \delta_2(\varepsilon)$.

By Lemma 3.3, the zero solution of (1) is uniformly integrally stable.

4. Uniformly Asymptotically Integral Stability

In this section, we will discuss the uniformly asymptotically integral stability using the comparison technique.

At first we give the difinition of the uniformly asymptotically integral stability.

〔Definition 4.1〕 The zero solution of the system (1) is said to be uniformly integrally attractive if for any $\varepsilon > 0$, $\alpha \geqq 0$ and $t_0 \geqq 0$ there exist $\gamma(\alpha, \varepsilon) > 0$ and $T(\alpha, \varepsilon) > 0$ such that the inequality $\|x_0\| < \alpha$ and $\int_{t_0}^{\infty} \sup_{\|x\| \leqq \beta} \|g(t, x)\| dt < \gamma(\alpha, \varepsilon)$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geqq t_0 + T(\alpha, \varepsilon)$.

〔Definition 4.2〕 The zero solution of the system (1) is said to be uniformly asymptotically integrally stable if it is uniformly integrally stable and is uniformly integrally attractive.

〔Definition 4.3〕 The zero solution of the system (1) is said to be uniformly stable if for any $\varepsilon > 0$ and $t_0 \geqq 0$ there exists $\delta(\varepsilon) > 0$ such that $\|x_0\| < \delta(\varepsilon)$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geqq t_0$.

〔Definition 4.4〕 The zero solution of the system (1) said to be uniformly attractive if for any $\varepsilon > 0$ and $t_0 \geqq 0$ there exist $\delta_0 > 0$ and $T(\varepsilon) > 0$ such that the inequality $\|x_0\| < \delta_0$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geqq t_0 + T(\varepsilon)$.

〔Definition 4.5〕 The zero solution of the system (1) is said to be uniformly asymptotically stable if it is uniformly stable and uniformly attractive.

We shall need the comparison theorem to prove uniformly asymptotically integrally stable. Let us state the comparison theorem in the following form.

〔Theorem 4.1〕 *Let* $V(t, x) \in C(I \times R^n, R^n)$ *and* $V(t, x)$ *be locally Lipschitzian in x. Assume the function* $V(t, x)$ *satisfies*

$$\dot{V}_{(1)}(t, x) \leqq h(t, V(t, x)), \text{ where } h(t, u) \in C(I \times R^+, R).$$

Let $r(t, t_0, x_0)$ *be a maximal solution of the scalar differential equation*

$$\frac{du}{dt} = h(t, u), \; u(t_0) = u_0,$$

existing to the right of t_0. *If* $x(t, t_0, x_0)$ *is any solution of the system (1) existing for all* $t \geqq t_0$ *such that* $V(t_0, x_0) \leqq u_0$, *then* $V(t, x(t, t_0, x_0)) \leqq r(t, t_0, u_0)$ *for all* $t \geqq t_0$.

For proof, see 〔6〕.

The definition of uniform stability of the zero solution of the system (1) can be formulated by means of monotonic function, as can be seen by the following.

〔Theorem 4.2〕 *The zero solution of the system (1) is uniformly stable if and only if there exists a function* $\rho(r)$, *where* $\rho(r)$ *is continuous, strictly monotone increasing and* $\rho(0) = 0$, *verifying the estimate* $\|x(t, t_0, x_0)\| \leqq \rho(\|x_0\|)$ *for all* $t \geqq t_0$.

We may likewise state the following theorem with respect to uniformly asymptotic stability.

〔Theorem 4.3〕 *The zero solution of the system (1) is uniformly asymptotically stable if and only if there* *exist functions $\rho(r)$ and $\sigma(r)$, where $\rho(r)$ is continuous, strictly monotone increasing and $\rho(0)=0$, $\sigma(r)$ is continous, decreasing, and $\sigma(t)\longrightarrow\infty$ as $t\longrightarrow\infty$, such that $\|x(t, t_0, x_0)\|\leqq\rho(\|x_0\|)\sigma(t-t_0)$ for all $t\geqq t_0$.*

To use the second method of Liapunov, which attempts to make statement about the stability properties directly by using suitable functions, we need the scalar differential equation

(6) $$\frac{du}{dt}=h(t, u),\ u(t_0)=u_0,$$

where $h(t, u)\in C(I\times R^+, R)$ and $h(t, 0)\equiv 0$.

〔Theorem 4.4〕 *Suppose that there exist functions $V(t, x)\in C(I\times R^n, R^+)$ and $h(t, u)\in C(I\times R^+, R)$ satisfying the following conditions :*

(i) *$a(t, \|x\|)\leqq V(t, x)$, $V(t, 0)\equiv 0$, where $a(t, r)$ is continuous in (t, r), nondecreasing in r for each t, $a(t, r)>0$ for any $r\neq 0$, $a(t, 0)\equiv 0$ and $a(t, r)\longrightarrow\infty$ as $r\longrightarrow\infty$ uniformly in t,*

(ii) *$|V(t, x)-V(t, y)|\leqq M\|x-y\|$, $M>0$,*

(iii) *$h(t, 0)\equiv 0$ and $h(t, u)$ is nonincreasing in u for each t,*

(iv) *$\dot V_{(1)}(t, x)\leqq h(t, V(t, x))$.*

Then the uniformly asymptotic stability of the zero solution of (6) implies the uniformly asymptotically integral stability of the zero solution of the system (1).

Proof. We first prove the uniformly integral stability. By Theorem 4.2, the unform stability of the zero solution of (6) implies the existence of a function $b(r)$ such that $b(r)$ is continuous, strictly increasing, $b(0)=0$ and $u(t, t_0, u_0)\leqq b(u_0)$ for all $t\geqq t_0$.

Now let $\alpha\geqq 0$ and $t_0\geqq 0$ be given, and let $\|x_0\|\leqq\alpha$. Then we have, from (ii), $V(t_0, x_0)\leqq M\alpha$. Let $x(t, t_0, x_0)$ be any solution of (1) with $\|x_0\|\leqq\alpha$, and let $m(t)\equiv V(t, x(t, t_0, x_0))-\varphi(t)$, where

$$\varphi(t)=M\int_{t_0}^{t}\|g(s, x(s, t_0, x_0))\|ds.$$

Using (iv), we get $\dot m(t)\leqq h(t, V(t, x(t, t_0, x_0)))$, from which it follows $\dot m(t)\leqq h(t, m(t))$, because of the nonincreasing property of $h(t, u)$ in u and the fact $m(t)\leqq V(t, x(t, t_0, x_0))$.

By the comparison theorem, we then have, as far as $x(t, t_0, x_0)$ exists to the right of t_0, $m(t)\leqq r(t, t_0, u_0)$, where $r(t, t_0, u_0)$ is the maximal solution of (6) with $u_0=m(t_0)$.

Let $\beta(\alpha)>0$ be so chosen that $b(M\alpha)+M\alpha<a(0, \beta(\alpha))$, this choice is clearly possible in view of the fact that $a(t, r)\longrightarrow\infty$ as $r\longrightarrow\infty$ uniformly in t.

We claim that, with this $\beta(\alpha)$, the zero solution of (1) is uniformly integrally stable whenever

$\|x_0\|\leqq\alpha$ and $\displaystyle\int_{t_0}^{\infty}\sup_{\|x\|\leqq\beta(\alpha)}\|g(t, x)\|dt<\alpha$.

Assuming that this claim is false, there exists $t^*>t_0$ such that $\|x(t^*, t_0, x_0)\|=\beta(\alpha)$ and $\|x(t, t_0, x_0)\|\leqq\beta(\alpha)$ for all $t\in[t_0, t^*]$. Then we obtain $a(0, \beta(\alpha))\leqq a(t^*, \|x(t^*, t_0, x_0)\|)$ $\leqq V(t^*, x(t^*, t_0, x_0))\leqq r(t^*, t_0, u_0)+\varphi(t^*)\leqq b(M\alpha)+M\alpha$, which contradicts.

Therefore, the zero solution of (1) is uniformly integrally stable.

Next, we prove that the zero solution of (1) is uniformly integrally attractive. By uniformly asymptotic stability of the zero solution of (6), we have $u(t, t_0, u_0)\leqq b(u_0)p(t-t_0)$ for all $t\geqq t_0$, where $b(r)$ is continuous, strictly increasing and $b(0)=0$, $p(t)$ is continuous, decreasing and $p(t)\longrightarrow\infty$ as $t\longrightarrow\infty$. If we are now given $\varepsilon>0$, $\alpha\geqq 0$ and $t_0\geqq 0$, we make the following choice: $M\gamma_1(\varepsilon)<a(0, \beta^{-1}(\varepsilon))$.

We put $\gamma(\alpha, \varepsilon)\equiv\min\{\gamma_1(\varepsilon), \alpha\}$. For any solution $x(t, t_0, x_0)$ of (1), we put

$m(t)\equiv V(t, x(t, t_0, x_0))-\varphi(t)$, where $\varphi(t)=M\displaystyle\int_{t_0}^{t}\|g(s, x(s, t_0, x_0))\|ds$.

As above seen, we hfave $m(t) \leqq r(t, t_0, u_0)$ for all $t \geqq t_0$, with $u_0 = m(t_0)$.

By the property of the function $p(t)$, there exists $T(\alpha, \varepsilon) > 0$ such that

$$p(t-t_0) < \frac{a(0, \beta^{-1}(\varepsilon)) - M\gamma(\alpha, \varepsilon)}{b(M\alpha)} \text{ for all } t \geqq t_0 + T(\alpha, \varepsilon).$$

For any solution $x(t, t_0, x_0)$ of (1) with $\|x_0\| \leqq \alpha$, there exists $t^* > t_0$ such that $\|x(t^*, t_0, x_0)\| < \beta^{-1}(\varepsilon)$.

In fact, if we assume that this is not true, then $\|x(t, t_0, x_0)\| \geqq \beta^{-1}(\varepsilon)$ for all $t \geqq t_0$. For all $t \geqq t_0 + T(\alpha, \varepsilon)$, we have

$$a(0, \beta^{-1}(\varepsilon)) \leqq a(t, \|x(t, t_0, x_0)\|) \leqq V(t, x(t, t_0, x_0))$$
$$\leqq b(u_0)p(t-t_0) + M\gamma(\alpha, \varepsilon) < a(0, \beta^{-1}(\varepsilon)),$$

which contradicts.

Consequently, there is t^* such that $\|x(t^*, t_0, x_0)\| < \beta^{-1}(\varepsilon)$ and $t_0 < t^* < t_0 + T(\alpha, \varepsilon)$.

Thus, by the uniformly integral stability of the zero solution of (1), we have $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geqq t_0 + T(\alpha, \varepsilon)$, provided $\|x_0\| \leqq \alpha$ and $\int_{t_0}^{\infty} \sup_{\|x\| \leqq \beta} \|g(t, x)\| dt < \alpha$.

This proves that the zero solution of (1) is uniformly asymptotically integrally stable.

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