The Integrability Tensor of Riemannian Submersions and Submanifolds

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Introduction

Let $\pi ; M \to M'$ be a Riemannian submersion. Let N' be a submanifold of M' and $N = \pi^{-1}(N')$. The relation between N and N' is studied by H. B. Lawson [3] and R. H. Escobales [1]. When $M = S^{2m+1}$ and $M' = CP_m(c)$ is studied by K. Yano and M. Kon [8], H. Naitoh and M. Takeuchi [4] and others.

In this paper we will study that the relation between N and N' under the condition of the integrability tensor A associated with the submersion.

1. Submersions

Let M and M' be Riemannian manifolds of dimensions m + p and m respectively. By a Riemannian submersion we mean a C^∞ mapping $\pi ; M \to M'$ such that π is of maximal rank and π_* preserves the lengths of horizontal vectors , i. e., vectors orthogonal to the fiber $\pi^{-1}(y)$ for some $y \in M'$.

Let X denote a tangent vector at $x \in M$. Then X decomposes as $\mathscr{V}X + \mathscr{H}X$, where $\mathscr{V}X$ is tangent to the fiber through x and $\mathscr{H}X$ is perpendicular to it. If $X = \mathscr{V}X$, it is called a vertical vector; and if $X = \mathscr{H}X$, it is called horizontal. Let $\tilde{\nabla}$ and $\tilde{\nabla}^*$ denote Riemannian connections of M and M' respectively.

We define a integrability tensor A associated with the submersion. For any vector fields E and F on M,

$$(1,1)\quad A_E F = \mathscr{V}\tilde{\nabla}_{\mathscr{H}E}(\mathscr{H}F) + \mathscr{H}\tilde{\nabla}_{\mathscr{H}E}(\mathscr{V}F).$$

A is a (1, 2)-tensor, and it has the following properties [6] :

(1) At each point, A_E is a skew-symmetric linear operator on the tangent space of M and it reverses the horizontal and vertical subspaces.

(2) $A_E = A_{\mathscr{H}E}$.

(3) For horizontal vector fields, A has the alternation property $A_X Y = -A_Y X$.

We define a vector field X on M to be basic provided X is horizontal and π-related to a vector field X_* on M' . Every vector field X_* on M' has a unique horizontal lift X to M and X is basic. And denote by $X = h.l.(X_*)$.

Throughout this paper, we assume that the fibers are totally geodesic in M.

LEMMA 1. [6]. If X and Y are basic vector fields on M, then

(1) $\tilde{g}(X, Y) = \tilde{g}^*(X_*, Y_*) \circ \pi$,

(2) $\mathscr{H}[X, Y]$ is the basic vector field corresponding to $[X_*, Y_*]$,

(3) $\mathscr{H}\tilde{\nabla}_X Y$ is the basic vector field corresponding to $\tilde{\nabla}^*_{X_*} Y_*$, where $\tilde{g}$ and $\tilde{g}^*$ are the metrics of M and M' respectizely.

LEMMA 2. [6]. Let X and Y be horizontal vector fields and V is vertical vector field on M. Then

(1) $\tilde{\nabla}_V X = \mathcal{H}\tilde{\nabla}_V X$,
(2) $\tilde{\nabla}_X V = A_X V + \mathcal{V}\tilde{\nabla}_X V$,
(3) $\tilde{\nabla}_X Y = \mathcal{H}\tilde{\nabla}_X Y + A_X Y$.

Furthermore, if X is basic, $\mathcal{H}\tilde{\nabla}_V X = A_X V$.

Denote by $\tilde{R}$ the curvature tensor of M. The horizontal lift of the curvature tensor $\tilde{R}^*$ of M′ will also be denoted by $\tilde{R}^*$, explicitly, if h_1, h_2, h_3, h_4 are horizontal tangent vectors to M, we set.

$$\tilde{g}(\tilde{R}^*_{h_1 h_2}(h_3), h_4) = \tilde{g}^*(\tilde{R}^*_{h_{1*} h_{2*}}(h_{3*}), h_{4*})\circ\pi$$

where $h_{i*} = \pi_*(h_i)$.

LAMMA 3. [6]. Let X, Y, Z and H be horizontal vector fields and V and W be vertical vector fields, then

(1) $\tilde{R}(X, V, Y, W) = \tilde{g}((\tilde{\nabla}_V A)_X Y, W) + \tilde{g}(A_X V, A_Y W)$,
(2) $\tilde{R}(X, Y, Z, V) = \tilde{g}((\tilde{\nabla}_Z A)_X Y, V)$,
(3) $\tilde{R}(X, Y, Z, H) = \tilde{R}^*(X, Y, Z, H) - 2\tilde{g}(A_X Y, A_Z H) + \tilde{g}(A_Y Z, A_X H) + \tilde{g}(A_Z X, A_Y H)$.

For horizontal vector fields X, Y, Z and H on M we set

$$\tilde{D}(X, Y, Z, H) = -2\tilde{g}(A_X Y, A_Z H) + \tilde{g}(A_Y Z, A_X H) + \tilde{g}(A_Z X, A_Y H).$$

PROPOSITION 4. Let $\pi: M \to M'$ be a Riemannian submersion with totally geodesic fibers. Then we obtain following equations

$$(\tilde{\nabla}_C \tilde{D})(X, Y, Z, H) = -2\tilde{g}((\tilde{\nabla}_C A)_X Y, A_Z H) - 2\tilde{g}(A_X Y, (\tilde{\nabla}_C A)_Z H) + \tilde{g}((\tilde{\nabla}_C A)_Y Z, A_X H) + \tilde{g}(A_Y Z, (\tilde{\nabla}_C A)_X H) + \tilde{g}((\tilde{\nabla}_C A)_Z X, A_Y H) + \tilde{g}(A_Z X, (\tilde{\nabla}_C A)_Y H)$$

where X, Y, Z, H and C are horizontal vector fields on M,

$$((\tilde{\nabla}^*_{C_*}\tilde{R}^*)(X_*, Y_*, Z_*, H_*)) = (\tilde{\nabla}_C \tilde{R})(X, Y, Z, H) + \tilde{R}(\mathcal{V}\tilde{\nabla}_C X, Y, Z, H) + \tilde{R}(X, \mathcal{V}\tilde{\nabla}_C Y, Z, H) + \tilde{R}(X, Y, \mathcal{V}\tilde{\nabla}_C Z, H) + \tilde{R}(X, Y, Z, \mathcal{V}\tilde{\nabla}_C H) - (\tilde{\nabla}_C \tilde{D})(X, Y, Z, H)$$

where X_*, Y_*, Z_*, H_* and C_* are tangent vector fields on M′ and X, Y, Z, H, C are their horizontal lifts. Especially if M is a space of constant curvature, we obtain

$$(\tilde{\nabla}_C \tilde{D})(X, Y, Z, H) = 0.$$
$$((\tilde{\nabla}^*_{C_*}\tilde{R}^*)(X_*, Y_*, Z_*, H_*))\circ\pi = (\tilde{\nabla}_C \tilde{R})(X, Y, Z, H) = 0.$$

Proof. From the following equations

$$C(\tilde{D}(X, Y, Z, H)) = -2\tilde{g}(\tilde{\nabla}_C(A_X Y), A_Z H) - 2\tilde{g}(A_X Y, \tilde{\nabla}_C(A_Z H)) + \tilde{g}(\tilde{\nabla}_C(A_Y Z), A_X H) + \tilde{g}(A_Y Z, \tilde{\nabla}_C(A_X H)) + \tilde{g}(\tilde{\nabla}_C(A_Z X), A_Y H) + \tilde{g}(A_Z X, \tilde{\nabla}_C(A_Y H)),$$

$$\tilde{\nabla}_C(A_X Y) = (\tilde{\nabla}_C A)_X Y + A_{\tilde{\nabla}_C X} Y + A_X(\tilde{\nabla}_C Y)$$

we obtain

$$(\tilde{\nabla}_C \tilde{D})(X, Y, Z, H) = C(\tilde{D}(X, Y, Z, H)) - \tilde{D}(\tilde{\nabla}_C X, Y, Z, H) - \tilde{D}(X, \tilde{\nabla}_C Y, Z, H) - \tilde{D}(X, Y, \tilde{\nabla}_C Z, H) - \tilde{D}(X, Y, Z, \tilde{\nabla}_C H)$$
$$= -2\tilde{g}((\tilde{\nabla}_C A)_X Y, A_Z H) - 2\tilde{g}(A_X Y, (\tilde{\nabla}_C A)_Z H) + \tilde{g}((\tilde{\nabla}_C A)_Y Z, A_X H) + \tilde{g}(A_Y Z, (\tilde{\nabla}_C A)_X H) + \tilde{g}((\tilde{\nabla}_C A)_Z X, A_Y H) + \tilde{g}(A_Z X, (\tilde{\nabla}_C A)_Y H).$$

Using LEMMA 1 and LEMMA 3 we see that

$$
\begin{aligned}
&((\tilde{\nabla}^*{}_{C*}\tilde{R}^*)(X_*, Y_*, Z_*, H_*))\circ\pi \\
&\quad = (C_*(\tilde{R}^*(X_*, Y_*, Z_*, H_*))\circ\pi - \tilde{R}^*(\tilde{\nabla}^*_{C*}X_*, Y_*, Z_*, H_*))\circ\pi \\
&\qquad - \tilde{R}^*(X_*, \tilde{\nabla}^*{}_{C*}Y_*, Z_*, H_*)\circ\pi \\
&\qquad - \tilde{R}^*(X_*, Y_*, \tilde{\nabla}^*{}_{C*}Z_*, H_*)\circ\pi - \tilde{R}^*(X_*, Y_*, Z_*, \tilde{\nabla}^*{}_{C*}H_*)\circ\pi \\
&\quad = C(\tilde{R}^*(X, Y, Z, H)) - \tilde{R}^*(\mathcal{H}\tilde{\nabla}_C X, Y, Z, H) - \tilde{R}^*(X, \mathcal{H}\tilde{\nabla}_C Y, Z, H) \\
&\qquad - \tilde{R}^*(X, Y, \mathcal{H}\tilde{\nabla}_C Z, H) - \tilde{R}^*(X, Y, Z, \mathcal{H}\tilde{\nabla}_C H) \\
&\quad = (\tilde{\nabla}_C\tilde{R})(X, Y, Z, H) + \tilde{R}(\mathcal{V}\tilde{\nabla}_C X, Y, Z, H) + \tilde{R}(X, \mathcal{V}\tilde{\nabla}_C Y, Z, H) \\
&\qquad + \tilde{R}(X, Y, \mathcal{V}\tilde{\nabla}_C Z, H) + \tilde{R}(X, Y, Z, \mathcal{V}\tilde{\nabla}_C H) - (\tilde{\nabla}_C\tilde{D})(X, Y, Z, H).
\end{aligned}
$$

When M is a space of constant curvature c, we have $\tilde{R}(X, Y)V = c(\tilde{g}(V, Y)X - \tilde{g}(V, X)Y) = 0$. From A_XY is vertical, by Lemma 3 (2), we obtain $\tilde{g}((\tilde{\nabla}_C A)Y, A_ZH) = \tilde{R}(X, Y, C, A_ZH) = 0$. Therefore we obtain $(\tilde{\nabla}_C\tilde{D})(X, Y, Z, H) = 0$. Since $\tilde{R}(X, Y)Z = c(\tilde{g}(Z, Y)X - \tilde{g}(Z, X)Y)$, we obtain $((\tilde{\nabla}^*_{C*}R^*)(X_*, Y_*, Z_*, H_*))\circ\pi = (\tilde{\nabla}_C\tilde{R})(X, Y, Z, H) = 0$.

Q. E. D.

2. Submanifolds

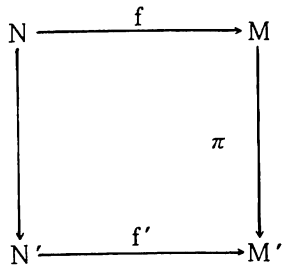
Let M be a $(2m+1)$-dimensional regular Sasakian manifold with structre tensors $(\phi, \xi, \eta, \tilde{g})$ such that there exists a fibering $\pi: M \to M/\xi = M'$, where M' denote the set of orbits of ξ and is a real $2m$-dimensional Kaehlerian manifold. This is a submersion with totally geodesic fibers. We denote by $(J, \tilde{g}^*)$ the Kaehlerian structure of M'. Then we have

$$\phi X = \mathrm{h.l.}(JX_*), \qquad \tilde{g}(X, Y) = \tilde{g}^*(X_*, Y_*)\circ\pi$$

for any vector fields X_* and Y_* on M'. Let $\tilde{\nabla}$ and $\tilde{\nabla}^*$ denote the Riemannian connection of M and M' respectivery.

Let N be an $(2n+1)$-dimensional submanifold tangent vector field ξ of M and N' be an $2n$-dimensional submanifold of M'. We assume that the diagram

(*)



commutes. Let ∇ and ∇^* denote the Riemannian connections of N and N' respectively.

For any vector fields X_* and Y_* of M', we have

$$\mathrm{h.l.}(\tilde{\nabla}^*{}_{X_*}Y_*) = \tilde{\nabla}_XY - \eta(\tilde{\nabla}_XY)\xi \quad [7]$$

By LEMMA 2. (3) and $\mathrm{h.l.}(\tilde{\nabla}^*{}_{X_*}Y_*) = \mathcal{H}(\tilde{\nabla}_XY)$, for any horizontal vector fields X and Y of M , we obtain

$$A_XY = \eta(\tilde{\nabla}_XY)\xi = \tilde{g}(\tilde{\nabla}_XY, \xi)\xi = \tilde{g}(Y, \phi X)\xi.$$

By LEMMA 2. (2) and $\tilde{\nabla}_X\xi = -\phi(X)$ and ϕX is horizontal vector field of M, we obtain

$$A_X\xi = -\phi(X) \qquad X \in \mathscr{H}(M).$$

A $(2n+1)$-dimensional submanifold N of M is said to be invariant if the structure vector field ξ is tangent to N everywhere on N and ϕX is tangent to N for any vector field X tangent to N a every point of N, that is, $\phi T_x(N) \subset T_x(N)$ for each $x \in N$. A n-dimensional submanifold N of M is said to be totally real if $\phi T_x(N) \subset T_x(N)^{\perp}$ for each $x \in N$. When $n = m + 1$ we have that the structure vector field ξ is tangent to N. We say that a submanifold N of M is generic if $\phi T_x(N)^{\perp} \subset T_x(N)$ for each $x \in N$.

We assume that submanifold N of M is invariant. Then, for any normal vector field B of N and horizontal vector field X of N, we obtain

$$A_B X = -A_X B = -\tilde{g}(B, \phi X)\xi = 0.$$

We assume that the submanifold N of M is totally real. Then, for any horizontal vector fields X and Y of N, we obtain

$$A_X Y = \tilde{g}(Y, \phi X)\xi = 0.$$

We assume that the submanifold N of M is generic. Then, for any normal vector fields B and C of N, we obtain

$$A_B C = \tilde{g}(C, \phi B)\xi = 0.$$

Let M and M′ be Riemannian manifolds of dimensions $m + p$ and m respectively. Suppose now that N is an $(n + p)$-dimensional submanifold of M which respects the submersion. That is, suppose there is a submersion $\pi; N \to N'$ where N′ is a submanifold of M′ such that the diagram

$$(**) \qquad \begin{array}{ccc} N & \xrightarrow{\ f\ } & M \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ N' & \xrightarrow{\ f'\ } & M' \end{array}$$

commutes and the immersion f is a diffeomorphism on the fibers. We assume that the fibers are totally geodesic in M.

Let h be second fundamental form of the submanifold N. Let h′ be second fundamental form of the submanifold N′. Let $g(\nabla)$ and $g^*(\nabla^*)$ denote the induced metrics (connections) of N and N′ respectively. Then the Gauss–Weingarten formuras are given by

(2.1) $\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$, $\tilde{\nabla}_X E = -S_E X + D_X E$, $X, Y \in \mathfrak{X}(N)$, $E \in \mathfrak{X}^{\perp}(N)$.

where $g(S_E X, Y) = g(h(X, Y), E)$ and D is the connection in the normal bundle $T(N)^{\perp}$. Note that the normal space is always horizontal. We set $C_E X = \mathcal{H} S_E \mathcal{H} X$ where X is tangent to N. Then we have the following equations

(2.2) $S_E X = C_E X + A_E X$,

(2.3) $S_E V = D_V E - \tilde{\nabla}_V E = D_V E - \mathcal{H}\tilde{\nabla}_V E$

where X and V are horizontal and vertical tangent vectors on N [1].

In the case $A_X Y = 0$ where X and Y are horizontal vector fields of N, we have following equations

(2.4) $\tilde{\nabla}_X Y = \mathcal{H}\tilde{\nabla}_X Y + A_X Y = \text{h. l. } \nabla^*_{X_*} Y_* + \text{h. l. } h'(X_*, Y_*)$.

on the other hand, by the Gauss equation

(2.5) $\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$.

Hence we obtain

(2.6) $\nabla_X Y = \text{h. l. } \nabla^*_{X_*} Y_*$, $\quad h(X, Y) = \text{h. l. } h'(X_*, {}_*Y)$.

For vertical vector field V of N and horizontal vector field X, by LEMMA 2 (2) and (2.1), we have

(2.7) $A_X V + \mathcal{V}\tilde{\nabla}_X V = h(X, V) + \nabla_X V$.

Since assumption $A_XY=0$, we obtain $g(A_XV, Y)=-g(V, A_XY)=0$ where for any horizontal vector field Y of N. Hence A_XV is normal to N. Thus, by (2.6), we obtain

(2.8) $\quad A_XV=h(X, V) \qquad \nabla_XV=\mathcal{V}\tilde{\nabla}_XV.$

By LEMMA 2

$$\tilde{\nabla}_VX=\mathcal{H}\tilde{\nabla}_VX=A_XV.$$

On the other hand, by (2, 8)

$$\tilde{\nabla}_VX=\nabla_VX+h(X, V)=\nabla_VX+A_XV.$$

Therefore we obtain

(2.9) $\quad \nabla_VX=0.$

In (**), we assume that dim(fibers) = 1. Since fibers are totally geodesic, we have

(2.10) $\quad \nabla_VV=0 \qquad h(V, V)=0.$

Let L_x be linear span of $h(T_x(N), T_x(N))$ and $L=\bigcup_{x\in N}L_x$.

For the second fundamental form h we define its covariant derivative $\overline{\nabla}h$ by

$$(\overline{\nabla}h)(X, Y, Z)=D_Xh(Y, Z)-h(\nabla_XY, Z)-h(Y, \nabla_XZ)$$

PROPOSITION 5. In (**), let M be a space of constant curvature and dim(M) = m+1 and dim(M′) = m and fibers be totally geodesic. We assume $A_XY=0$ where X and Y are horizontal vector fields of N, and $A_BC=0$ where C is normal vector of N and $B\in L$. Then h is parallel if and only if h′ is parallel.

Proof. Let X, Y and Z be horizontal vector fields and V be vertical vector field of N. By (2.8), (2.9) and (2.10), we obtain

(a) $\quad (\overline{\nabla}h)(V, V, V)=D_Vh(V, V)-2h(\nabla_VV, V)=0$

(b) $\quad (\overline{\nabla}h)(X, V, V)=D_Xh(V, V)-2h(\nabla_XV, V)=-2h(\mathcal{V}\tilde{\nabla}_XV, V)=0.$

Since $A_BC=0$, we obtain $\tilde{g}(A_{h(X,Y)}V, C)=-\tilde{g}(V, A_{h(X,Y)}C)=0$. Therefore $A_{h(X,Y)}V$ is tangent to N. On the other hand, we have following equations

$$\tilde{\nabla}_Vh(X, Y)=-S_{h(X,Y)}V+D_Vh(X, Y)$$

$$\tilde{\nabla}_Vh(X, Y)=A_{h(X,Y)}V.$$

Thus, we have

(2, 11) $\quad D_Vh(X, Y)=0.$

By (2.9) we have

(c) $\quad (\overline{\nabla}h)(V, X, Y)=D_Vh(X, Y)-h(\nabla_VX, Y)-h(X, \nabla_VY)=D_Vh(X, Y)=0.$

Let D′ be the connection in the normal bundle $T(N')^{\perp}$.

By (2.6) and $D_Xh(Y, Z)=h.l.\,D'_{X_*}h'(Y_*, Z_*)$, we have

(d) $$\begin{aligned}(\overline{\nabla}h)(X, Y, Z)&=D_Xh(Y, Z)-h(\nabla_XY, Z)-h(Y, \nabla_XZ)\\&=h.l.\,(D'_{X_*}h'(Y_*, Z_*)-h'(\nabla^*_{X_*}Y_*, Z_*)\\&\quad-h'(Y_*, \nabla^*_{X_*}Z_*))\\&=h.l.\,(\nabla^*h')(X_*, Y_*, Z_*).\end{aligned}$$

Since M is a space of constant curvature, $\overline{\nabla}h$ is symmetric trilinear in virture of the Codazzi equation. Hence we have our assertion. Q. E. D.

Let $\pi: S^{2m+1}\to CP_m(c)$ be the standard Riemannian submersion from a sphere [6]. Let J be the complex structure tensor on $CP_m(c)$ and $(S^{2m+1}, \phi, \xi, \eta, \tilde{g})$ be a standard Sasakian manifold.

COROLLARY 6 [4]. Let N′ be a totally real submanifold of $CP_m(c)$, we set $N=\pi^{-1}(N')$. Assume that the linear span $N^1{}_y(N')$ of $h'(T_y(N'), T_y(N'))$ is containd in $J(T_y(N'))$ for each $y\in N'$. Then h is parallel if and only if h′is parallel.

Proof. N is a totally real submanifold of S^{2m+1} if and only if N′ is a totally real submanifold of $CP_m(c)$ [8]. Therefore $A_X Y = 0$ where X and Y is horizontal vector fields of N. For horizontal vector field X, we have h. l. $(JX_*) = \phi(X)$. The assumption $N^{\perp}{}_y(N') \subset J(T_y(N'))$ implies that $Jh(X_*, Y_*)$ is tangent to N′. And hence, by (2.6), $\phi(h(X, Y))$ is tangent to N. Therefore, for normal vector field C of N and $B \in L$, $A_B C = \tilde{g}(C, \phi B)\xi = 0$. Q. E. D.

Let N be a submanifold of a Riemannian manifold M. For every $x \in N$, let ρ_x ; M → M denote the involutive isometry with the initial data $\rho_x(x) = x$ and $(\rho_x)_*(X + B) = -X + B$ ($X \in T_x(N)$, $B \in T_x(N)^{\perp}$). If we have $\rho_x(N) = N$ for all $x \in N$, then we call N a symmetric submanifold of M. If we have a neighborhood U and $\rho_x(U) = U$ for all $x \in N$, then we call N a locally symmetric submanifold of M.

LEMMA 7. [5]. Let N be a submanifold of Riemannian manifold M. If N is a locally symmetric submanifold, then the second fundamental form of N is parallel.

LEMMA 8. [5]. Let N be a submanifold of Riemannian symmetric space M. Then N is a locally symmetric submanifold if and only if N has the following two conditions ;

(1) the second fundamental form of N is parallel,

(2) for each point $x \in N$, there exists totally geodesic submanifold P of M satisfying $x \in P$ and $T_x(P) = T_x(N)^{\perp}$.

LEMMA 9. [5]. Let M be a Riemannian symmetric space and a space of constant curvature. Then the submanifold N of M is a locally symmetric if and only if the second fundamental form of N is parallel.

Using those LEMMAs and PROPOSITION 4, 5 we obtain following theorem.

THEOREM 10. In (* *), let M be complete and a space of constant curvature c and dim (M) = m + 1 and dim (M′) = m and fibers be totally geodesic. We assume $A_X Y = 0$ where X and Y are horizontal vector fields of N, and $A_B C = 0$ where B and C is normal vector fields of N. Then N is a locally symmetric submanifold if and only if N′ is a locally symmetric submanifold.

Proof. If N′ is a locally symmetric submanifold, by LEMMA 7, the second fundamental from of N′ is parallel. Since PROPOSITION 5, the second fundamental form of N is parallel. From M is a space of constant curvature, M is a Riemannian locally symmetric ($\tilde{\nabla}\tilde{R} = 0$). From those and M is complete, M is Riemannian symmetric space. Therfore, by LEMMA 9, N is a locally symmetric submanifold.

If N is a locally symmetric submanifold, the second fundamental form of N is parallel. Therfore, by PROPOSITION 5, the second fundamental form of N′ is parallel. From PROPOSITION 4, M′ is a Riemannian locally symmetric space. If π ; M → M′ is a submersion and M is complete, then M′ and the fibers are also complete. Therefore M′ is a Riemannian symmetric space. Since $A_B C = 0$ and M is a space of constant curvature c, we have

$$\tilde{g}(\tilde{R}^*(B_*, C_*)D_*, X_*) \circ \pi = \tilde{g}(\tilde{R}^*(B, C)D, X) = \tilde{R}^*(X, D, B, C)$$
$$= \tilde{R}(X, D, B, C) - 2\tilde{g}(A_X D, A_B C) + \tilde{g}(A_D B, A_X C) + \tilde{g}(A_B X, A_D C)$$
$$= -\tilde{g}(\tilde{R}(B, C)X, D) = -\tilde{g}(c(\tilde{g}(X, C)B - \tilde{g}(X, B)), D) = 0,$$

where B_*, C_* and D_* are normal vector fields of N′ and X_* is tangent vectore field of N′. Therefore we obtain $\widetilde{R}^*(T_y(N')^{\perp}, T_y(N')^{\perp})T_y(N')^{\perp} \subset T_y(N')^{\perp}$. Therefore there exists totally geodesic submanifold P such that $T_y(P)=T_y(N')^{\perp}$. Since LEMMA 8, N′ is a locally symmetric submanifold.

Q. E. D.

COROLLARY 11. N′ be a generic submanifold of a complex projective space CP_m with constant holomorphic sectional curvature 4. If the second fundamental form of N′ is parallel, then N′ is a locally symmetric submanifold of CP_m.

Proof. In (*), we set $M=S^{2m+1}$ and $M'=CP_m$. By [9], if the second fundamental form of N′ is parallel, then N′ is a totally real. From the second fundamental form of N is parallel, N is a locally symmetric submanifold of S^{2m+1}! By THEOREM 10, N′ is a locally symmetric submanifold of CP_m.

Q. E. D.

Let $e_1, \ldots\ldots, e_{n+q}$ be an orthonormal basis in $T_x(N)$. The mean curvature vector μ of N is defined to be $\mu=\frac{1}{n+p}$ (Trh), which is independent of the choice of a basis.

PROPOSITION 12. In (**), let $\dim(M)=m+1$ and $\dim(M')=m$ and fibers be totally geodesic. We assume $A_BC=0$ where C is normal vector field of N and $B \in L$. Then the mean curvature vector μ of N is parallel if and only if the mean curvature vector μ' of N′ is parallel.

Proof. If we take an orthonormal basis $\{e_{i*}\}$ in $T_y(N')$. Then $\{e_i, v\}$ formes an orthonormal basis in $T_x(N)$ $(\pi(x)=y)$, where $v=V/g(V,V)$. By the definition of the mean curvature vector and (2 . 10), we have

$$\text{h. l.}(\mu')=\frac{1}{n}\text{h. l.}(\text{Trh}')=\frac{1}{n}\text{h. l.}\left(\sum_{i=1}^{n}h'(e_i, e_i)\right)$$
$$=\frac{1}{n}\left(\sum_{i=1}^{n}h(e_i, e_i)+h(v, v)\right)=\frac{1}{n}\text{Trh}=\frac{n+1}{n}\mu.$$

By the equation $D_Xh(Y,Z)=\text{h. l. } D'_{X*}h'(Y_*, Z_*)$, we obtain

$$\frac{n}{n+1}\text{h. l.}(D'_{X*}\mu')=D_X\mu.$$

By (2 . 11), we obtain

$$D_V\mu=D_V\left(\frac{1}{n+1}\text{Trh}\right)=\frac{1}{n+1}\sum_{i=1}^{n}D_Vh(e_i, e_i)=0.$$

Q. E. D.

We now define the curvature tensor $R^{\perp}$ of the normal bundle of N by

(2 , 12) $R^{\perp}(X,Y)B=D_XD_YB-D_YD_XB-D_{[X,Y]}B$

where X and Y are tangent vector fields on N and B is normal vector field on N. If $R^{\perp}$ vanishes identically, then the normal connection of N is said to be flat.

In (**), let $R^{\perp}$ and $R'^{\perp}$ (D and D′) denote the curvature tensors (the connections) of normal bundle of N and N′ respectively. We have the following Weigarten formulas;

$$\widetilde{\nabla}^*_{X*}B_*=-S'_{B*}X_*+D'_{X*}B_*, \qquad \widetilde{\nabla}_XB=-S_BX+D_XB,$$

where X_* is tangent vector field on N′ and B_* is normal vector field on N′ and h. l. $(X_*)=X$, h. l. $(B_*)=B$. On the other hand, we have h. l. $(\widetilde{\nabla}^*_{X*}B_*)=\mathscr{H}\widetilde{\nabla}_XB=\widetilde{\nabla}_XB-A_XB$.

Thus we have

(2.13) $h.l.(S'_{B*}X_*) = S_B X + A_X B$ and $h.l.(D'_{X*}B_*) = D_X B$.

Let X and Y be horizontal vector fields on N and B and C be normal vector fields on N. Then (2.13) implies

$$h.l.(D'_{X*}D'_{Y*}B_*) = D_X D_Y B,\quad h.l.(D'_{Y*}D'_{X*}B_*) = D_Y D_X B.$$

Since $h.l.[X_*, Y_*] = [X, Y] - 2A_X Y$, we find

$$h.l.(D'_{[X*, Y*]}B_*) = D_{[X,Y]}B - 2D_{A_X Y}B.$$

From these equations we have

(2, 14) $\tilde{g}^*(R'^{\perp}(X_*, Y_*)B_*, C_*)\circ\pi = \tilde{g}(R^{\perp}(X, Y)B, C) + 2\tilde{g}(D_{A_X Y}B, C)$.

PROPOSITION 13. In (**), let fibers be totally geodesic. We assume $A_B C=0$ where B and C is normal vector fields on N. Then the normal connection of N is flat if and only if the normal connection of N′ is flat.

Proof. Let X and Y be horizontal vector fields on N and V be vertical vector field on N and B and C be normal vector fields on N. Then, by $\tilde{g}(A_B V, C) = -\tilde{g}(V, A_B C) = 0$, $A_B V$ is tangent to N. Frome the following equations;

$$\tilde{\nabla}_V B = -S_B V + D_V B \text{ and } \tilde{\nabla}_V B = \mathscr{H}\tilde{\nabla}_V B = A_B V,$$

we obtain $D_V B=0$. From $A_X Y$ is vertical, we have

(2.15) $\tilde{g}^*(R'^{\perp}(X_*, Y_*)B_*, C_*) = \tilde{g}(R^{\perp}(X, Y)B, C)$.

The equation implies that if the normal connection of N is flat then the normal connection of N′ is flat. Since $D_V B=0$, we obtain $D_V D_X B=0$ and $D_X D_V B=0$. Since $[X,V]$ is vertical, we have $D_{[X,V]}B=0$. Therefore we obtain $R^{\perp}(X,V)B=0$. From the integrability of the vertical distribution, for vertical vector fields V and W, $[V, W]$ is vertical. Therefore we obtain $R^{\perp}(V, W)B=0$. Thus, by (2.15), if the normal connection of N′ is flat then the normal connection of N is flat. Q. E. D.

EXAMPLE [8]. Let $\pi; S^{2m+1} \to CP_m(c)$ be the standard Riemannian submersion. We set $N = S^1(r_1)\times\ldots\ldots\times S^1(r_{m+1})$ where $r_1^2 + \ldots\ldots + r_{m+1}^2 = 1$. Then N is a generic and a totally real submanifold. These are $A_B C=0$ and $A_X Y=0$. Here N has parallel second fundamental form and flat normal connection.

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