On the Uniformly Ultimate Boundedness of Solutions

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1. Introduction

It is well-known that A. M. Liapunov has discussed the stability of solutions of the system of ordinary differential equations by utilizing a scalar function satisfying some conditions. For the boundedness as well as the stability, Liapunov's second method is very useful and powerful theory. Its usefulness and power lie in the fact that the criterion of the boundedness can be decided from the differential equations without any knowledge of their solutions.

However, it is great difficult to find the Liapunov function satisfying certain conditions. Therefore, it is important to obtain a weak sufficient condition for the boundedness theorem. Many authors have discussed the boundedness. (cf. [1], [2], [3], [4], [5], [6], [7], [8].)

In the previous papers [9], [10], [11], [12], [13], we obtained weak sufficient conditions for the various types of boundedness.

The purpose of this paper is to give some extension of the sufficient conditions for the uniformly ultimate boundedness of solutions of a system of ordinary differential equations.

2. Definitions and Notations

First, we summarize some basic definitions and notations we will need later on.

Let I denote the interval $0 \leq t < \infty$ and R^n denote the Euclidean n-space. For $x \in R^n$ let $\|x\|$ be any norm of x. Let S_K denote the set of x such that $\|x\| < K$, $K > 0$, $\overline{S}_K$ be the closure of S_K, and S_K^c be the complement of S_K. We shall denote by $C(I \times R^n \times R^m, R^K)$ the set of all continuous functions defined on $I \times R^n \times R^m$ with values in R^K.

We consider a system of differential equations.

$$(1) \qquad \frac{dx}{dt} = f(t, x),$$

where x is an n-vector and $f(t, x) \in C(I \times R^n, R^n)$ is an n-vector function.

Suppose that $f(t, x)$ is smooth enough to ensure existence, uniqueness and continuous dependence of the solutions of the initial value problem associated with (1).

Throughout this paper a solution through a point $(t_0, x_0) \in I \times R^n$ will be denoted by such a form as $x(t, t_0, x_0)$.

We introduce the following definitions.

〔Definition 1〕

The solutions of (1) are equi-bounded, if for any $\alpha > 0$ and any $t_0 \in I$, there exists a $\beta(t_0, \alpha) > 0$ such that $\|x_0\| \leq \alpha$ implies $\|x(t, t_0, x_0)\| < \beta(t_0, \alpha)$ for all $t \geq t_0$.

〔Definition 2〕

The solutions of (1) are uniformly bounded, if the β in Definition 1 is independent of t_0.

〔Definition 3〕

The solutions of (1) are equi-ultimately bounded for bound B, if there exists a $B>0$ and if corresponding to any $\alpha>0$ and $t_0 \in I$, there exists a $T(t_0, \alpha)>0$ such that $\|x_0\| \leq \alpha$ implies that $\|x(t, t_0, x_0)\| < B$ for all $t \geq t_0+T(t_0, \alpha)$.

〔Definition 4〕

The solutions of (1) are uniformly ultimately bounded for bound B, if the T in Definition 3 is independent of t_0.

〔Definition 5〕

Corresponding to a continuous scalar function $V(t, x)$ defined on an open set, we define the function.

$$\dot{V}_{(1)}(t, x) = \overline{\lim_{h\to 0+}} \frac{1}{h} \{V(t+h, x+hf(t, x)) - V(t, x)\}.$$

In case $V(t, x)$ has continuous partial derivatives of the first order, it is evident that

$$\dot{V}_{(1)}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t, x),$$

where "・" denote a scalar product.

3. Preliminary Results

In some cases, the following theorem on boundedness is more convenient to apply. We consider a system.

$$(2) \quad \frac{dx}{dt} = f(t, x, y), \qquad \frac{dy}{dt} = g(t, x, y),$$

where $f(t, x, y) \in C(I\times R^n \times R^m, R^n)$ and $g(t, x, y) \in C(I\times R^n\times R^m, R^m)$.

T. Yoshizawa gave a sufficient condition for the uniform boundedness and the uniformly ultimately boundedness. Here, we repeat those theorems.

【Theorem 3.1】

Suppose that there exists a Liapunov function $V(t, x, y)$ *defined on* $0\leq t<\infty$, $\|x\|+\|y\|\geq K$, *where K can be large, which satisfies the following conditions;*

(i) $V(t, x, y)$ *tends to infinity uniformly for* (t, x) *as* $\|y\|\to\infty$,

(ii) $V(t, x, y)\leq b(\|x\|, \|y\|)$, *where* $b(r, s)$ *is continuous,*

(iii) $\dot{V}_{(2)}(t, x, y)\leqq 0$.

Moreover, suppose that corresponding to each $M>0$ *there exists a Liapunov function* $W(t, x, y)$ *defined on* $0\leq t<\infty$, $\|x\|\geq K_1(M)$, $\|y\|\leq M$, *where* K_1 *can be large, which satisfies the following conditions;*

(iv) $W(t, x, y)$ *tends to infinity uniformly for* (t, y) *as* $\|x\|\to\infty$,

(v) $W(t, x, y)\leq c(\|x\|)$, *where* $c(r)$ *is contnuous,*

(vi) $\dot{W}_{(2)}(t, x, y)\leqq 0$.

Then the solutions of (2) *are uniformly bounded.*

【Theorem 3.2】

For the system (2), *assume that there exists a Liapunov function* $V(t, x, y)$ *defined on* $0\leq t<\infty$, $\|x\|<\infty$, $\|y\|\geq K>0$, *which satsfies the following conditions;*

(i) $a(\|y\|) \leq V(t, x, y) \leq b(\|y\|)$, *where* $a(r)$ *and* $b(r)$ *are continuous, increasing and* $a(r) \to \infty$ *as* $r \to \infty$,

(ii) $\dot{V}_{(2)}(t, x, y) \leq -c(\|y\|)$, *where* $c(r) > 0$ *is continuous.*

Suppose that corresponding to each M, there exists a Liapunov function $W(t, x, y)$ *defined on* $0 \leq t < \infty$, $\|x\| \geq K_1(M)$, $\|y\| \leq M$, *which satisfies the following conditions*;

(iii) $a_1(\|x\|) \leq W(t, x, y) \leq b_1(\|x\|)$, *where* $a_1(r)$ *and* $b_1(r)$ *are continuous and* $a_1(r) \to \infty$ *as* $r \to \infty$,

(iv) $\dot{W}_{(2)}(t, x, y) \leq 0$.

Moreover, assume that letting B *be such that* $b(K) < a(B)$, *there exists a Liapunov function* $U(t, x, y)$ *defined on* $T \leq t < \infty$, $\|x\| \geq K_2 \geq 0$, $\|y\| \leq B$, *which satisfies the following conditions*;

(v) $a_2(\|x\|) \leq U(t, x, y) \leq b_2(\|x\|)$, *where* $a_2(r)$ *and* $b_2(r)$ *are continuous and increasing,*

(vi) $\dot{U}_{(2)}(t, x, y) \leq -c_2(\|x\|)$, *where* $c_2(r) > 0$ *is continuous.*

Then the solutions of (2) *are uniformly ultimately bounded.*

For the proof of these theorems, see [1].

4. Result

【Theorem 4.1】

For the system (2), *assume that there exists a Liapunov function* $V(t, x, y)$ *defined on* $I \times R^n \times S_K^c$, *where* $K > 0$ *can be large, which satisfies the following conditions*:

(i) $a_1(t, \|y\|) \leqq V(t, x, y) \leqq b_1(\|y\|)$, *where* $a_1(t, r)$ *is continuous in* (t, r), $a_1(t, r) > 0$ *for any* $r \neq 0$, $a_1(t, r) \equiv 0$, *nondecreasing with respect to t for each fixed r and* $a_1(t, r) \longrightarrow \infty$ *as* $r \longrightarrow \infty$ *uniformly in t,* $b_1(r)$ *is continuous and increasing,*

(ii) $\dot{V}_{(2)}(t, x, y) \leqq -c_1(t, \|y\|)$, *where* $c_1(t, r) > 0$ *is continuous in* (t, r).

Suppose that corresponding to each $L > 0$, *there exists a Liapunov function* $W(t, x, y)$ *defined on* $I \times S_{H(L)}^c \times \overline{S}_L$, *which satisfies the following conditions*:

(iii) $a_2(t, \|x\|) \leqq W(t, x, y) \leqq b_2(\|x\|)$, *where* $a_2(t, r)$ *is continuous in* (t, r), $a_2(t, r) > 0$ *for any* $r \neq 0$, $a_2(t, 0) \equiv 0$, *nondecreasing with respect to t for any fixed r and* $a_2(t, r) \longrightarrow \infty$ *as* $r \longrightarrow \infty$ *uniformly in t,* $b_2(r)$ *is continuous,*

(iv) $\dot{W}_{(2)}(t, x, y) \leqq 0$.

Moreover, assume that $C > 0$ *be such that* $b_1(K) < a_1(0, C)$, *there exists a Liapunov function* $U(t, x, y)$ *defined on* $T \leqq t < \infty$, $\|x\| \geqq H^* > 0$, $\|y\| \leqq C$, *which satisfies the following conditions*:

(v) $a_3(t, \|x\|) \leqq U(t, x, y) \leqq b_3(\|x\|)$, *where* $a_3(t, r)$ *is continuous in* (t, r), $a_3(t, r) > 0$ *for any* $r \neq 0$, $a_3(t, 0) \equiv 0$ *and nondecreasing with respect to t and r,* $b_3(r)$ *is continuous and increasing,*

(vi) $\dot{U}_{(2)}(t, x, y) \leqq -c_3(t, \|x\|)$, *where* $c_3(t, r) > 0$ *is continuous in* (t, r).

Then the solutions of (2) *are uniformly ultimately bounded.*

Proof. For any $\alpha > 0$ such that $K < \alpha$, consider a solution $\{x(t, t_o, x_o, y_o), y(t, t_o, x_o, y_o)\}$ of a system (2), where $t_o \in I$, $\|x_o\| \leqq \alpha$ and $\|y_o\| \leqq \alpha$. Choose a $\beta(\alpha) > 0$ so large that $b_1(\alpha) < a_1(0, \beta(\alpha))$.

First of all, we prove that $\|y(t, t_o, x_o, y_o)\| < \beta(\alpha)$ for all $t \geqq t_o$. Suppose that this is not true, there is a t_1 such that $\|y(t_1, t_o, x_o, y_o)\| \leqq \beta(\alpha)$ and $t_1 > t_o$. Then there exist t_2 and t_3, $t_o \leqq t_2 < t_3 \leqq t_1$, such that $\|y(t_2, t_o, x_o, y_o)\| = \alpha$, $\|y(t_3, t_o, x_o, y_o)\| = \beta(\alpha)$ and $\alpha < \|y(t, t_o, x_o, y_o)\| < \beta(\alpha)$ for $t_2 < t < t_3$. Consider the function

$V(t, x(t, t_o, x_o, y_o), y(t, t_o, x_o, y_o))$ on $t_2 \leq t \leq t_3$. Then we have

$$a_1(0, \beta(\alpha)) \leq a_1(t_3, \|y(t_3, t_o, x_o, y_o)\|) \leq V(t_3, x(t_3), y(t_3)) \leq V(t_2, x(t_2), y(t_2)) \leq b_1(\|y(t_2, t_o, x_o, y_o)\|) \leq b_1(\alpha),$$

where $x(t_K) = x(t_K, t_o, x_o, y_o)$ and $y(t_K) = y(t_K, t_o, x_o, y_o)$, $k = 2, 3$, which contradicts $b_1(\alpha) < a_1(0, \beta(\alpha))$. Therefore $\|y(t, t_o, x_o, y_o)\| < \beta(\alpha)$ for all $t \geq t_o$.

We put $\eta(\alpha) = \max\{\alpha, H(\beta(\alpha))\}$ and consider a Liapunov function $W(t, x, y)$ defined on $I \times S^c_{H(\beta(\alpha))} \times \overline{S}_{\beta(\alpha)}$. Choose a $\beta^*(\alpha) > 0$ so large that $b_2(\eta(\alpha)) < a_2(0, \beta^*(\alpha))$. We shall show that $\|x(t, t_o, x_o, y_o)\| < \beta^*(\alpha)$ for all $t \geq t_o$. Suppose that it is not so, there is a t_1 such that $\|x(t_1, t_o, x_o, y_o)\| \geq \beta^*(\alpha)$ and $t_1 > t_o$. Then there exist t_2 and t_3, $t_o \leq t_2 < t_3 \leq t_1$, such that $\|x(t_2, t_o, x_o, y_o)\| = \eta(\alpha)$, $\|x(t_3, t_o, x_o, y_o)\| = \beta^*(\alpha)$ and $\eta(\alpha) < \|x(t, t_o, x_o, y_o)\| < \beta^*(\alpha)$ for $t_2 < t < t_3$.

Because $\|y(t, t_o, x_o, y_o)\| < \beta(\alpha)$ for all $t \geq t_o$, we have $\|y(t, t_o, x_o, y_o)\| < \beta(\alpha)$ for $t_2 \leq t \leq t_3$. Consider the function $W(t, x(t, t_o, x_o, y_o), y(t, t_o, x_o, y_o))$ on $t_2 \leq t \leq t_3$.

Because the function $W(t, x(t, t_o, x_o, y_o), y(t, t_o, x_o, y_o))$ is nondecreasing in t,

$$a_2(0, \beta^*(\alpha)) \leq a_2(t_3, \|x(t_3, t_o, x_o, y_o)\|) \leq W(t_3, x(t_3), y(t_3)) \leq W(t_2, x(t_2), y(t_2)) \leq b_2(\eta(\alpha)),$$

which contradicts $b_2(\eta(\alpha)) < a_2(0, \beta^*(\alpha))$.

Therefore, for any $\alpha > 0$ and $t_o \in I$, there exist $\beta^*(\alpha) > 0$ and $\beta(\alpha) > 0$ such that $\|x_o\| \leq \alpha$, $\|y_o\| \leq \alpha$ implies $\|x(t, t_o, x_o, y_o)\| < \beta^*(\alpha)$ and $\|y(t, t_o, x_o, y_o)\| < \beta(\alpha)$ for all $t \geq t_o$. This means that the solutions of (2) are uniformly bounded.

Next we shall prove that there is a $t_1 > t_o$ such that $\|y(t_1, t_o, x_o, y_o)\| < K$. In fact, if we assume that this is not true, then $\|y(t, t_o, x_o, y_o)\| \geq K$ for all $t \geq t_o$. By (ii), there exists a $\gamma(\alpha) > 0$ such that if $\|x\| < \infty$ and $K \leq \|y\| \leq \beta(\alpha)$, $\dot{V}_{(2)}(t, x, y) \leq -\gamma(\alpha)$. Therefore we have $V(t, x(t, t_o, x_o, y_o), y(t, t_o, x_o, y_o)) \leq V(t_o, x_o, y_o) - \gamma(\alpha)(t - t_o)$. If $t > t_o + T_1(\alpha)$, where $T_1(\alpha) = (b_1(\alpha) - a_1(0, K))/\gamma(\alpha)$,

$$\begin{aligned} a_1(0, K) \leq V(t, x(t, t_o, x_o, y_o), y(t, t_o, x_o, y_o)) &\leq V(t_o, x_o, y_o) - \gamma(\alpha)(t - t_o) \\ &< b_1(\alpha) - b_1(\alpha) + a_1(0, K) \\ &= a_1(0, K), \end{aligned}$$

and hence there arise a contradiction. Consequently, there is a t_1 such that $\|y(t_1, t_o, x_o, y_o)\| < K$ and $t_o \leq t_1 < t_o + T_1(\alpha)$. By the choice of C, we can see that $\|y(t, t_o, x_o, y_o)\| < C$ for all $t \geq t_1$. Thus, by the uniformly boundedness of solutions of (2), we have $\|y(t, t_o, x_o, y_o)\| < C$ for all $t \geq t_o + T_1(\alpha)$.

Now we put $K^* = \max\{C, H^*\}$. Then, if $\|x_o^*\| \leq K^*$ and $\|y_o^*\| \leq K^*$,

(3) $\quad \|x(t, t_o, x_o^*, y_o^*)\| < \beta^*(K^*)$ for all $t \geq t_o$.

As was seen above, $\|y(t, t_o, x_o, y_o)\| < C$ for all $t \geq t_o + T_1(\alpha)$, and consequently $\|y(t, t_o, x_o, y_o)\| < C$ for all $t \geq t_o + T_1(\alpha) + T$. Suppose that $\|x(t, t_o, x_o, y_o)\| \geq K^*$ for all $t \geq t_o + T_1(\alpha) + T$. For $K^* \leq \|x\| \leq \beta^*(\alpha)$, $\|y\| \leq C$ and $t \geq T$, by (vi), there exists a $\gamma^*(\alpha) > 0$ such that $\dot{U}_{(2)}(t, x, y) \leq -\gamma^*(\alpha)$, and hence

$$\begin{aligned} &U(t, x(t, t_o, x_o, y_o), y(t, t_o, x_o, y_o)) \\ &\leq U(t_o + T_1(\alpha) + T, x(t_o + T_1(\alpha) + T, t_o, x_o, y_o), y(t_o + T_1(\alpha) + T, t_o, x_o, y_o)) \\ &\quad - \gamma^*(\alpha)(t - t_o - T_1(\alpha) - T). \end{aligned}$$

If $t > t_o + T_1(\alpha) + T + T_2(\alpha)$, where $T_2(\alpha) = \dfrac{1}{\gamma^*(\alpha)}\{b_3(\beta^*(\alpha)) - a_3(0, K^*)\}$,

$$\begin{aligned} a_3(0, K^*) \leq U(t, x(t, t_o, x_o, y_o), y(t, t_o, x_o, y_o)) &< b_3(\beta^*(\alpha)) - b_3(\beta^*(\alpha)) + a_3(0, K^*) \\ &= a_3(0, K^*), \end{aligned}$$

we have a contradiction. Therefore, there exists a t^* such that $\|x(t^*, t_o, x_o, y_o)\| < K^*$ and $t_o + T_1(\alpha) + T \leq t^* < t_o + T_1(\alpha) + T_2(\alpha) + T$, and hence, by (3), $\|x(t, t_o, x_o, y_o)\| < \beta^*(K^*)$

for all $t \geqq t^*$, because $\|x(t^*, t_0, x_0, y_0)\| < K^*$, $\|y(t^*, t_0, x_0, y_0)\| < K^*$ and $x(t, t^*, x(t^*), y(t^*)) = x(t, t_0, x_0, y_0)$. Thus we have $\|x(t, t_0, x_0, y_0)\| < \beta^*(K^*)$ for all $t \geqq t_0 + T_1(\alpha) + T_2(\alpha) + T$ and $\|y(t, t_0, x_0, y_0)\| < C$ for all $t \geqq t_0 + T_1(\alpha)$.

If we put $B = \beta^*(K^*)$ and $T^*(\alpha) = T_1(\alpha) + T_2(\alpha) + T$, clearly $C \leqq B$ and $T_1(\alpha) \leqq T^*(\alpha)$.

Hence, for any $\alpha > 0$ and any $t_0 \in I$, if $\|x_0\| \leqq \alpha$, $\|y_0\| \leqq \alpha$, we have a B and a $T^*(\alpha)$ such that $\|x(t, t_0, x_0, y_0)\| < B$ and $\|y(t, t_0, x_0, y_0)\| < B$ for all $t \geqq t_0 + T^*(\alpha)$, where B is clearly a positive constant independent of particular solutions and $T^*(\alpha)$ is a positive constant depending only on $\alpha > 0$. This proves that the solutions of (2) are uniformly ultimately bounded.

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