On the Uniformly Totally Asymptotic Stability of a System of Ordinary Differential Equations

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1. Introduction

Consider a system of differential equations

$$\frac{dx}{dt} = f(t,x). \qquad (1)$$

We can assume without loss of generality that $f(t,0)\equiv 0$.

In addition to (1), let a perturbed differential equations

$$\frac{dx}{dt} = g(t,x). \qquad (2)$$

If a system (2) is considered as the equations of motion for a real, physical system upon which certain small perturbation forces act, described by $g(t,x)-f(t,x)$, it must be realized that often such forces are not accurately known ; in general, there are at most estimates for the perturbations. Thus, the assumption $g(t,0)\equiv 0$ is an idealization not justified by physical reality. If this assumption is dropped, the stability statement that small perturbation forces $g(t,x)-f(t,x)$ produce only small derivations cannot be formulated by means of the usual stability concept. Then an extension of this concept, called the total stability, is needed.

In the Soviet terminology this type of stability is called stability under constantly acting perturbations, which was introduced by Dubošin [1].

Many authors have discussed the total stability [2], [3], [4], [5], [6].

By introducing the limiting equations, Armado D'anna[9] showed the uniformly total stability and uniformly totally asymptotic stability of a bounded solution of an almost periodic equation.

In this paper, we will state the uniformly totally asymptotic stability of the zero solution of a general system of differential equations by using a Liapunov function.

2. Definitions and Notations

Let I denote the interval $0\leqslant t<\infty$ and R^n denote Euclidean n-space. For $x\in R^n$, let $\|x\|$ be the Euclidean norm of x, and we shall denote by S_H the set of x such that $\|x\|\leqslant H$, $H>0$.

Let us consider a system of the equations (1) with $f\in C(I\times W, R^n)$ where W is an open subset of R^n and $C(I\times W, R^n)$ denotes the collection of all continuous functions defined on $I\times W$ with values in R^n.

The function f is said to be admissible if for every $(t_0, x_0)\in I\times W$, there exists a unique noncontinuable solution $x(t, t_0, x_0, f)$ of (1) satisfying the initial condition $x(t_0, t_0, x_0, f)=x_0$.

If for any compact set $K\subset R^n$, there exists a constant $L(K)>0$ such that $\|f(t,x)-f(t,x')\|\leqslant L(K)\|x-x'\|$ for $x\in K$, $x'\in K$, we shall write $f(t,x)\in\overline{C}_0(x)$.

We introduce the following definitions.

[Definition 1]

The zero solution of (1) is said to be uniformly stable if for any $\varepsilon>0$ and any $t_0\in I$ there exists a $\delta(\varepsilon)>0$ such that the inequality $\|x_0\|<\delta(\varepsilon)$ implies $\|x(t,t_0,x_0,f)\|<\varepsilon$ for all $t\geqslant t_0$.

[Definition 2]

The zero solution of (1) is said to be uniformly attractive if there exist a $\delta_o > 0$ and a $T(\varepsilon) > 0$ for any $\varepsilon > 0$ such that if $\| x_o \| < \delta_o$, $\| x(t, t_o, x_o, f) \| < \varepsilon$ for all $t \geqq t_o + T(\varepsilon)$.

[Definition 3]

The zero solution of (1) is said to be uniformly asymptotically stable if it is uniformly stable and uniformly attractive.

[Definition 4]

The zero solution of (1) is said to be uniformly stable if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon)$ such that if $(t_o, x_o) \in I \times S_\delta$, and if $g \in C(I \times W, R^n)$ and satisfies $\| g(t,x) - f(t,x) \| < \delta$ for $t \geqq t_o$, $\| x \| < \varepsilon$, then any solution $x(t, t_o, x_o, g)$ of the system (2) satisfies $\| x(t, t_o, x_o, g) \| < \varepsilon$ for all $t \geqq t_o$.

[Definition 5]

The zero solution of (1) is said to be uniformly totally attractive if there are two numbers θ and ε_o, $0 < \theta < \varepsilon_o$, and for each $\gamma > 0$, there exists a $T(\gamma) > 0$ and a $\beta(\gamma) > 0$ such that if $(t_o, x_o) \in I \times S_\theta$, and if $g \in C(I \times W, R^n)$ and satisfies $\| g(t,x) - f(t,x) \| < \beta(\gamma)$ for $t \geqq t_o$, $\| x \| \leqq \varepsilon_o$, then any solution $x(t, t_o, x_o, g)$ of the system (2) satisfies $\| x(t, t_o, x_o, g) \| < \gamma$ for all $t \geqq t_o + T$.

[Definition 6]

The zero solution of (1) is said to be uniformly totally asymptotically stable, if it is uniformly totally stable and is uniformly totally attractive.

[Definition 7]

Corresponding to a continuous scalar function $V(t,x)$ defined on an open set, we define the function

$$V'_{(1)}(t,x) = \overline{\lim_{h \to 0+}} \frac{1}{h} \{ V(t+h,\ x + hf(t,x)) - V(t,x) \}.$$

In case $V(t,x)$ has continuous partial derivatives of the first order, it is evident that

$$V'_{(1)}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t,x),$$

where " • " denotes a scalar product.

3. Preliminary Results

【 Theorem 1 】

Suppose that $f(t, x)$ *of* (1) *is continuous on* $I \times S_H$ *and* $f(t, 0) \equiv 0$。
If $f(t, x) \in \overline{C}_o(x)$ *and if the solution of* (1) *is uniformly asympototically stable, then it is totally stable.*

For proof of this theorem, see [5].

【 Theorem 2 】

Suppose that $f(t, x)$ *of* (1) *is continuous on* $I \times S_H$ *and that* $f(t, 0) \equiv 0$.
If $f(t, x) \in \overline{C}_o(x)$ *and if there exists a Liapunov function* $V(t, x)$ *defined on* $I \times S_H$, *which satisfies the following conditions*;

(i) $V(t,0) \equiv 0$ *and* $V(t, x)$ *is continuous in* (t, x),

(ii) $a(t, \| x \|) \leqq V(t, x) \leqq b(\| x \|)$, *where the function* $a(t, r)$ *is continuous in* (t, r) *on* $I \times I$, $a(t,0) \equiv 0$, $a(t,r) > 0$ *for* $r \neq 0$ *and increases monotonically with respect to t and* r, *and* $b(r)$ *is a continuous increasing, positive function,*

(iii) $V'_{(1)}(t, x) \leqq -cV(t,x)$, *where* $c > 0$ *is a constant,*

(iv) $|V(t,x)-V(t,x')|\leqq K\|x-x'\|$, *where* $K>0$ *is a constant.*

Then the zero solution of the system (1) *is totally stable.*

For proof of this theorem, see [8].

4. Main Result

【 Theorem 3 】

Suppose that $f(t,x)$ *of* (1) *and* $g(t,x)$ *of* (2) *are admissible.*

If there exists a Liapunov function $V(t, x)$ *which is satisfies the following conditions;*

(i) $V(t,0)\equiv 0$ *and* $V(t,x)\in C(I\times S_H, R)$,

(ii) $a(t,\|x\|)\leqslant V(t,x)\leqslant b(\|x\|)$, *where the function* $a(t,r)$ *is continuous in* (t,r) *on* $I\times I$, $a(t,0)\equiv 0$, $a(t,r)>0$ *for* $r\neq 0$ *and nondecreases monotonically with respect to* t *and* r, *and* $b(r)$ *is a continuous increasing, positive function,*

(iii) $V'_{(1)}(t,x)\leqslant -\alpha V(t,x)$, *where* $\alpha>0$ *is a constant,*

(iv) $|V(t,x)-V(t,x')|\leqslant K\|x-x'\|$, *where* $K>0$ *is a constant.*

Then the zero solution of (1) *is uniformly totally asymptotically stable.*

[Proof]

We first show that the zero solution of (1) is uniformly asymptotically stable.

For any $\varepsilon>0$, choose a $\delta_1(\varepsilon)>0$ so that $b(\delta_1(\varepsilon))<a(0,\varepsilon)$, $0<\delta_1(\varepsilon)<\varepsilon$. Choose $\delta(\varepsilon)>0$ so small that $\alpha a(0,\delta_1(\varepsilon))-K\delta(\varepsilon)>0$. For any $(t_0,x_0)\in I\times S_{\delta(\varepsilon)}$ and any $g(t,x)\in C(I\times W,R^n)$ which satisfies $\|g(t,x)-f(t,x)\|<\delta(\varepsilon)$ for any $t\in(t_0,\infty)$, $\|x\|<\varepsilon$, we suppose that a solution $x(t,t_0,x_0,g)$ of (2) satisfies $\|x(t^*,t_0,x_0,g)\|>\varepsilon$ at some $t^*, t^*\geqslant t_0$. Then there are t_1 and t_2 such that $t_0\leqslant t_1\leqslant t_2\leqslant t^*$, $\|x(t_1,t_0,x_0,g)\|=\delta_1(\varepsilon)$, $\|x(t_2,t_0,t_0,g)\|=\varepsilon$ and that $\delta_1(\varepsilon)<\|x(t,t_0,x_0,g)\|<\varepsilon$ for $t\in(t_1,t_2)$.

By (iii), we have

$$
\begin{aligned}
V'(t,x(t,t_0,x_0,g)) &< -\alpha V(t,x(t,t_0,x_0,g))+K\delta(\varepsilon)\\
&< -\alpha a(t,\|x(t,t_0,x_0,g)\|)+K\delta(\varepsilon)\\
&< -\alpha a(0,\delta_1(\varepsilon))+K\delta(\varepsilon)<0.
\end{aligned}
$$

Therefore $V(t,x)$ is monotone decreasing along $x(t,t_0,x_0,g)$.

From this, we have

$$
\begin{aligned}
a(0,\varepsilon) = a(0,\|x(t_2,t_0,x_0,g)\|) &< a(t_2,\|x(t_2,t_0,x_0,g)\|)\\
&< V(t_2,x(t_2,t_0,x_0,g)) < V(t_1,x(t_1,t_0,x_0,g))\\
&< b(\|x(t_1,t_0,x_0,g)\|)=b(\delta_1(\varepsilon)),
\end{aligned}
$$

which contradicts $b(\delta_1(\varepsilon))<a(0,\varepsilon)$.

Therefore, $\|x(t,t_0,x_0,g)\|<\varepsilon$ for all $t\geqslant t_0$, which shows that the zero solution of (1) is uniformly totally stable.

Next, we show that the zero solution of (1) is uniformly totally attractive.

Since the zero solution of (1) is uniformly totally stable, for any $\varepsilon>0$ and any $t_0\in I$, there exists a $\delta_0=\delta_0(\varepsilon)>0$, $0<\delta_0<\varepsilon$, such that if $\|x_0\|<\delta_0(\varepsilon)$, and if $g\in C(I\times W,R^n)$ and satisfies $\|g(t,x)-f(t,x)\|<\delta_0(\varepsilon)$ for $t\geqslant t_0$, $\|x\|<\varepsilon$, then any solution $x(t,t_0,x_0,g)$ of (2) satisfies $\|x(t,t_0,x_0,g)\|<\varepsilon$ for all $t\geqslant t_0$. We choose a $\beta=\beta(\varepsilon)>0$ sufficiently small so that $a(0,\delta_0)-K\beta>0$ and $b(\delta_0)>a(0,\delta_0)-K\beta$. Let (t_0,x_0) be an arbitrary point in $I\times S_{\delta_0}$ and $g(t,x)\in C(I\times W,R^n)$ be an arbitrary function such that $\|g(t,x)-f(t,x)\|<\alpha\beta$ for $t\geqslant t_0$, $\|x\|\leqslant\varepsilon$.

Let $T(\varepsilon)$ be such that $T(\varepsilon)=\dfrac{1}{\alpha}\log\dfrac{b(\delta_0)}{a(0,\delta_0)-K\beta}$.

Now we suppose that there is not a $t^*\in(t_0,t_0+T)$ such that $\|x(t^*,t_0,x_0,g)\|<\delta_0(\varepsilon)$.

Then we have $\| x(t, t_0, x_0, g) \| > \delta_0(\varepsilon)$ for all $t \in [t_0, t_0 + T]$. By(iii), for any $t \in [t_0, T]$, we have

$$V'(t, x(t, t_0, x_0, g)) \leqslant -\alpha V(t, x(t, t_0, x_0, g)) + K \| g(t, x) - f(t, x) \|$$
$$\leqslant -\alpha V(t, x(t, t_0, x_0, g)) + K\alpha\beta.$$

Hence we have

$$e^{\alpha t}\{V'(t, x(t, t_0, x_0, g)) + \alpha V(t, x(t, t_0, x_0, g))\} < K\beta\alpha e^{\alpha t},$$

or $\{e^{\alpha t} V(t, x(t, t_0, x_0, g))\}' \leqslant K\beta\alpha e^{\alpha t}$.

Therefore we have

$$e^{\alpha t} V(t, x(t, t_0, x_0, g)) \leqslant e^{\alpha t_0} V(t_0, x_0) + K\beta(e^{\alpha t} - e^{\alpha t_0})$$
$$\leqslant e^{\alpha t_0} V(t_0, x_0) + K\beta e^{\alpha t}.$$

Since this inequality is true for $t = t_0 + T$, we have

$$V(t_0 + T, x(t_0 + T, t_0, x_0, g)) \leqslant e^{-\alpha T} V(t_0, x_0) + K\beta$$
$$= \frac{a(0, \delta_0) - K\beta}{b(\delta_0)} b(\| x_0 \|) + K\delta_0$$
$$< \frac{a(0, \delta_0) - K\delta_0}{b(\delta_0)} b(\delta_0) + K\delta_0 = a(0, \delta_0(\varepsilon)).$$

Therefore we have

$$a(0, \delta_0(\varepsilon)) < a(t_0 + T, \| x(t_0 + T, t_0, x_0, g) \|)$$
$$< V(t_0 + T, x(t_0 + T, t_0, x_0, g))$$
$$< a(0, \delta(\varepsilon)).$$

This is a contradiction.

Thus we know that there exists a $t^* \in (t_0, t_0 + T)$ such that $\| x(t^*, t_0, x_0, g) \| < \delta_0(\varepsilon)$.

From above, we have $\| x(t, t_0, x_0, g) \| < \varepsilon$ for all $t \geqslant t^*$ and $\| x(t, t_0, x_0, g) \| < \varepsilon$ for all $t \geqslant t_0 + T(\varepsilon)$. This implies that the zero solution of (1) is uniformly totally attractive.

Thus we see that the zero solution of (1) is uniformly totally asymptotically stable.

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