On the Partially Total Stability and the Partially Total Boundedness of a System of Ordinary Differential Equations

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I. Introduction

Suppose that the families of motions are defined by a system of differential equations

$$\frac{dz}{dt} = F(t, z). \tag{1}$$

We can assume without loss of generality that $F(t, 0) \equiv 0$.
In addition to (1), let a perturbed differential equation

$$\frac{dw}{dt} = F(t, w) + G(t, w). \tag{2}$$

In general the term $G(t, w)$ describes the effect of small disturbances caused by friction, current heat, etc. It must be realized that often such disturbances are not accurately known ; in general, there are at most estimates for the perturbations. In particular the assumption $G(t, 0) \equiv 0$ has no justification whatsoever. But its omission denies the hypothesis that (2) even admits the solution $z \equiv 0$.

Therefore we introduce the concept of total stability which is the extention of the usual stability. In the Soviet terminology this type of stability is called stability under constantly acting perturbations, which was introduced by Dubošin [1].

Many authors have discussed the total stability [2], [3], [4], [5], [6].

We also stated some extentions of the sufficient conditions for the total stability and the total boundedness in the previous paper [7].

In many applications, we need to see the qualities not of the whole solution but of the partial.

In this paper, we describe several results concerning the total stability and total boundedness with respect to a part of the solutions of differential equations.

2. Total Stability and Total Boundedness

Let I denote the interval $0 \leqq t < \infty$ and R^p denote Euclidean p-space. For $z \in R$, let $\|z\|$ be the Euclidean norm of z, and we shall denote by S_H the set of such that $\|z\| \leqq H$, $H > 0$.

We consider a system of differential equations (1), where z is a p-vector and $F(t, z)$ is a p-vector function which is defined on a region in $I \times R^p$.

Throughout this paper a solution through a point (t_0, z_0) in $I \times R^p$ will be denoted by such a form as $z(t\,;\,z_0, t_0)$. If for any compact set $K \subset R^p$, there exists a constant $L(K) > 0$ such that $\|F(t, z) - F(t, z')\| \leqq L(K)\|z - z'\|$ for $z \in K$, $z' \in K$, we shall write $F(t, z) \in C_0(z)$.

We introduce the following definitions.

〔Definition 1〕

The zero solution of (1) is said to be uniformly stable if for any $\varepsilon > 0$ and any $t_0 \in I$ there exists a $\delta(\varepsilon) > 0$ such that the inequality $\|z_0\| < \delta$ implies $\|z(t\,;\,z_0, t_0)\| < \varepsilon$ for all $t \geqq t_0$.

〔Definition 2〕

The zero solution of (1) is said to be uniform-asymptotically stable if it is uniformly stable and there exist $\delta_0>0$ and $T(\varepsilon)>0$ for any $\varepsilon>0$ such that if $\|z_0\|<\delta_0$, $\|z(t\ ;\ z_0,t_0)\|<\varepsilon$ for all $t\geqq t_0+T(\varepsilon)$.

〔Definition 3〕

The zero solution of (1) is totally stable, if given $\varepsilon>0$, there exists a $\delta(\varepsilon)>0$ such that for any equation (2), where $\|G(t,z)\|<\delta$, the solution $z(t\ ;\ z_0,\ t_0)$ of (2) satisfies $\|z(t\ ;z_0,\ t_0)\|<\varepsilon$ for any initial value such that $\|z_0\|<\delta$, $t\geqq t_0\geqq 0$.

〔Definition 4〕

The solutions of (1) are totally bounded, if given $\alpha>0$, there exist two numbers $\beta(\alpha)>0$, $\gamma(\alpha)>0$ such that if $z_0\in S_\alpha$, then $\|z(t\ ;\ z_0,t_0)\|<\beta(\alpha)$ for all $t\geqq t_0$, where $z(t\ ;\ z_0,\ t_0)$ is the solution of (2) in which $\|G(t,z)\|<\gamma(\alpha)$, provided $\alpha<\|z\|<\beta$.

〔Definition 5〕

Corresponding to a continuous scalar function $V(t,z)$ defined on an open set, we define the function

$$V'_{(1)}(t,z)=\overline{\lim_{h\to 0+}}\frac{1}{h}\{V(t+h,\ x+hF(t,z))-V(t,z)\}.$$

In case $V(t,z)$ has continuous partial derivatives of the first order, it is evident that

$$V'_{(1)}(t,z)=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial z}\cdot F(t,z),$$

where " · " denotes a scalar product.

〔Theorem 1〕

Suppose that $F(t,z)$ *of* (1) *is continuous on* $I\times S_H$ *and* $F(t,0)\equiv 0$.

If $F(t,z)\in \overline{C}_0(z)$ *and if the solution of* (1) *is uniform-asymptotically stable, then it is totally stable.*

〔Theorem 2〕

Suppose that $F(t,z)$ *of* (1) *is continuous on* $I\times R^P$ *and that there exists a continuous function* $V(t,z)$ *defined on* $D:0\leqq t<\infty$, $\|z\|\geqq L$, *where* L *may be large, which satisfies the following conditions* ;

(i) $a(\|z\|)\leqq V(t,z)\leqq b(\|z\|)$, *where* $a(r)$ *and* $b(r)$ *are continuous increasing and* $a(r)\to\infty$ as $r\to\infty$,

(ii) $V(t,z)\in\overline{C}_0(z)$, *and* $V'_{(1)}(t,z)\leqq -c(\|z\|)$, *where* $c(r)>0$ *is continuous.*

Then, the solutions of (1) *are totally bounded.*

For proofs of these theorems, see reference 〔4〕.

〔Theorem 3〕

Suppose that $F(t,z)$ *of* (1) *is continuous on* $I\times S_H$ *and that* $F(t,0)\equiv 0$.

If $F(t,z)\in\overline{C}_0(z)$ *and if there exists a function* $V(t,z)$ *defined on* $I\times S_H$, *which satisfies the following conditions* ;

(i) $V(t,0)\equiv 0$ *and* $V(t,z)$ *is continuous in* (t,z),

(ii) $a(t,\|z\|)\leqq V(t,z)\leqq b(\|z\|)$, *where the function* $a(t,r)$ *is continuous in* (t,r) *on* $I\times I$, $a(t,0)\equiv 0$, $a(t,r)>0$ *for* $r\neq 0$ *and increases monotonically with respect to* t *and* r, *and* $b(r)$ *is a continuous increasing, positive function,*

(iii) $V'_{(1)}(t,z)\leqq -cV(t,z)$, *where a* $c>0$ *is a constant.*

(iv) $|V(t,z)-V(t,z')|\leqq K\cdot\|z-z'\|$, *where* $K>0$ *is a constant.*

Then the zero solution of the system (1) *is totally stable.*

〔Theorem 4〕

Suppose that $F(t,z)$ *of* (1) *is continuous on* $I\times R^P$ *and that there exists a continuous function* $V(t,z)>0$

defined on D : $0 \leqq t < \infty$, $\| z \| \geqq L$, *where L may be large, which satisfies the following conditions* ;

(i) $a(t, \| z \|) \leqq V(t, z) \leqq b(\| z \|)$, *where the function* $a(t,r)$ *is continuous in* (t,r) *and* $a(t,r) \to \infty$, *uniformly in* t *as* $r \to \infty$,

(ii) $V(t, z) \in \overline{C}_0(z)$,

(iii) $V'_{(1)}(t, z) \leqq -c(t, \| z \|)$, *where the function* $c(t,r)$ *is continuous in* (t,r) *and there exists* $k(\alpha,\beta)>0$ *such that* $c(t,r) > k(\alpha,\beta)$ *for* $\alpha \leqq r \leqq \beta$.

Then the solutions of (1) *are totally bounded.*

For proofs of these theorems, see reference〔7〕

3. Partially Total Stability and Partially Total Boundedness

We shall denote by $C(I \times R^m \times R^n, R^k)$ the set of all continuous functions defined on $I \times R^m \times R^n$ with values in R^k.

Consider a system of differential equations

$$\frac{d}{dt}(x, y) = (f_1(t, x, y),\ f_2(t, x, y)) \tag{3}$$

and a perturbed system

$$\frac{d}{dt}(u, v) = (f_1(t, u, v),\ f_2(t, u, v)) + (g_1(t, u, v),\ g_2(t, u, v)). \tag{4}$$

These are the cases that, in systems (1) and (2),

$z = (x, y) \in R^m \times R^n$,

$w = (u, v) \in R^m \times R^n$,

$F(t, z) = (f_1(t, x, y),\ f_2(t, x, y))$ and

$G(t, w) = (g_1(t, u, v),\ g_2(t, u, v))$

where $f_1, g_1 \in C(I \times R^m \times R^n, R^m)$ and $f_2, g_2 \in C(I \times R^m \times R^n, R^n)$.

Let $w(t\ ;\ w_0, t_0) = (u(t\ ;\ w_0, t_0),\ v(t\ ;\ w_0, t_0))$ be the solution of (4) starting from $w_0 = (u_0, v_0)$ at t_0.

We introduce the following definitions.

〔Definition 6〕

The zero solution of (3) is partially totally stable, if given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for any equation (4), where $\| G(t, w) \| < \delta$, the solution $w(t\ ;\ w_0, t_0)$ satisfies $\| u(t\ ;\ w_0, t_0) \| < \varepsilon$ for any initial value such that $\| w_0 \| < \delta$, $t \geqq t_0 \geqq 0$.

〔Definition 7〕

The solutions of (3) are partially totally bounded, if given $\alpha > 0$, there exist two numbers $\beta(\alpha), \gamma(\alpha) > 0$ such that if $\| w_0 \| < \alpha$, then $\| u(t\ ;\ w_0, t_0) \| < \beta(\alpha)$ for all $t \geqq t_0$, where $w(t\ ;\ w_0, t_0)$ is the solution of (4) in which $\| G(t, w) \| < \gamma(\alpha)$, provided $\alpha < \| w \| < \beta$.

〔Definition 8〕

Corresponding to a continuous scalar function $V(t, x, y)$ defined on an open set, we define the function

$$V'_{(3)}(t, x, y) = \overline{\lim_{h \to +0}} \frac{1}{h}\{V(t+h,\ x + hf_1(t, x, y),\ y + hf_2(t, x, y)) - V(t, x, y)\}.$$

In case $V(t, x, y)$ has continuous partial derivatives of the first order, it is evident that

$$V'_{(3)}(t, x, y) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f_1(t, x, y) + \frac{\partial V}{\partial y} \cdot f_2(t, x, y).$$

〔Theorem 5〕

Suppose that $F(t,z)$ *of* (3) *is continuous on* $I\times S_H$ *and that* $F(t,0)\equiv 0$.

If $F(t,z)\in \overline{C}_0(z)$ *and if there exists a continuous function* $V(t,x,y)$ *defined on* $I\times D\times R^n$ (D *is a domain in* R^m), *which satisfies the following conditions* ;

(i) $a(t,\|x\|)\leqq V(t,x,y)\leqq b(\|x\|)$, *where the function* $a(t,r)$ *is continuous in* (t,r) *on* $I\times I$, $a(t,0)\equiv 0$, $a(t,r)>0$ *for* $r\neq 0$ *and is nondecreasing with respect to* t *and* r, *and* $b(r)$ *is a continuous increasing, positive function and* $b(0)=0$,

(ii) $V'_{(3)}(t,x)\leqq -cV(t,x,y)$, *where a* c *is a positive constant.*

(iii) $|V(t,x,y)-V(t,x',y')|<K\|z-z'\|$, *where* $z=(x,y)$, $z'=(x',y')$ *and a K is a positive constant,*

then the zero solution of the system (3) *is partially totally stable.*

〔Proof〕 For any $\varepsilon>0$, choose a $\delta_1(\varepsilon)>0$ so that $b(\delta_1)<a(0,\varepsilon)$, $0<\delta_1<\varepsilon$ and also a $\delta(\varepsilon)>0$ so small that $\delta<\dfrac{ca(0,\delta_1)}{K}$, $0<\delta<\delta_1$. Suppose that a solution

$$w(t\,;\,w_0,t_0)=(u(t\,;\,w_0,t_0),\ v(t\,;\,w_0,t_0))$$

of (4), where $\|w_0\|<\delta$ and $\|G(t,w)\|<\delta$, satisfies $\|u(t\,;\,w_0,t_0)\|=\varepsilon$ at some t.

Then there are t_1 and t_2 such that

$$\|u(t_1\,;\,w_0,t_0)\|=\delta_1,\quad \|u(t_2\,;\,w_0,t_0)\|=\varepsilon$$

and $\delta_1<\|u(t\,;\,w_0,t_0)\|<\varepsilon$ for $t\in(t_1,t_2)$.

On the other hand, for $t\in[t_1,t_2]$

$$\begin{aligned}V'_{(4)}(t,u,v)&\leqq -cV_{(3)}(t,u,v)+K\|G(t,w)\|\\&\leqq -ca(t,u)+K\|G(t,w)\|\\&\leqq -ca(0,\delta_1)+K\delta<0.\end{aligned}$$

Thus, we have

$$V(t_1,u(t_1\,;\,w_0,t_0),\ v(t_1\,;\,w_0,t_0))>V(t_2,\ u(t_2\,;\,w_0,t_0),\ v(t_2\,;\,w_0,t_0)).$$

Therefore, we have

$$\begin{aligned}a(0,\varepsilon)&<a(t_2,\varepsilon)=a(t_2,\|u(t_2\,;\,w_0,t_0)\|)\\&\leqq V(t_2,\ u(t_2\,;\,w_0,t_0),\ v(t_2\,;\,w_0,t_0))<V(t_1,\ u(t_1\,;\,w_0,t_0),\ v(t_1\,;\,w_0,t_0))\\&\leqq b(\|u(t_1\,;\,w_0,t_0)\|)=b(\delta_1)<a(0,\varepsilon)\end{aligned}$$

which is a contradiction.

Therefore, $\|u(t\,;\,w_0,t_0)\|<\varepsilon$ for all $t\geqq t_0$, which shows the zero solution of (3) is partially totally stable.

〔Theorem 6〕

Suppose that $F(t,z)$ *of* (3) *is continuous on* $I\times R^n\times R^m$ and that there exists a continuous function $V(t,x,y)>0$ *defined on* $D=\{(t,x,y)\,;\,t\in I,\ \|x\|\geqq L,\ y\in R^m\}$, *where* L *may be large, which satisfies the following conditions* ;

(i) $a(t,\|x\|)\leqq V(t,x,y)\leqq b(\|x\|)$, *where the function* $a(t,r)$ *is a continuous nondecreasing function in* (t,r) *and* $a(t,r)\to\infty$ *uniformly in* t *as* $r\to\infty$,

(ii) $V(t,z)\in \bar{C}_0(z)$, *where* $V(t,z)=V(t,x,y)$,

(iii) $V'_{(3)}(t,x,y)\leqq -c(t,\|x\|)$, *where the function* $c(t,r)$ *is continuous in* (t,r) *and there exists* $k(\alpha,\beta)>0$ *such that* $c(t,r)>k(\alpha,\beta)$ *for* $\alpha\leqq r\leqq\beta$.

Then the solutions of (3) *are partially totally bounded.*

〔Proof〕 For any $\alpha>0$, choose a $\beta(\alpha)>0$ so that $a(0,\beta)>b(\alpha)$.

By (ii), for z such that $\alpha\leqq\|z\|$, $\|z'\|\leqq\beta$, there is a $K(\alpha)>0$ such that

$| V(t,z) - V(t,z') | \leqq K \| z - z' \|$.

Suppose that a solution $w(t\ ;\ w_0, t_0)$ of (4), where $\| w_0 \| < \alpha$ and $\| G(t,w) \| < \gamma(\alpha)$, satisfies $\| u(t\ ;\ w_0, t_0) \| = \beta(\alpha)$ at some t. Then there are t_1 and t_2 such that

$$\| u(t_1\ ;\ w_0, t_0) \| = \alpha. \quad \| u(t_2\ ;\ w_0, t_0) \| = \beta(\alpha)$$

and that $\alpha < \| u(t\ ;\ w_0, t_0) \| < \beta(\alpha)$ for $t \in (t_1, t_2)$.

In the domain $Q = \{(t,u,v) \mid 0 \leqq t < \infty,\ \alpha < \| u \| < \beta,\ v \in R^m\}$,

$$\begin{aligned} V'_{(4)}(t,u,v) &\leqq V'_{(3)}(t,u,v) + K \| G(t,w) \| \\ &\leqq -c(t,u) + K \| G(t,w) \| . \end{aligned}$$

Therefore, if we choose a $\gamma(\alpha)$ so that $\gamma(\alpha) \leqq \frac{k}{K}$, we have $V'_{(4)}(t,u,v) \leqq 0$ for $\| G(t,w) \| < \gamma(\alpha)$ in Q. Thus we have

$$\begin{aligned} a(0,\beta) < a(t_2,\beta) &= a(t_2, \| u(t_2\ ;\ w_0, t_0) \|) \\ &\leqq V(t_2, u(t_2\ ;\ w_0, t_0), v(t_2\ ;\ w_0, t_0)) \leqq V(t_1, u(t_1\ ;\ w_0, t_0), v(t_1\ ;\ w_0, t_0)) \\ &< b(\| u(t_1\ ;\ w_0, t_0) \|) = b(\alpha), \end{aligned}$$

which contradicts $a(0,\beta) > b(\alpha)$.

Therefore, we have $\| u(t\ ;\ w_0, t_0) \| < \beta(\alpha)$ for all $t \geqq t_0$, which shows that the solutions of (3) are partially totally bounded.

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