On Boundedness and Partial Boundedness of Solutions of a System of Ordinary Differential Equations

Masamichi ASO, Miki KUDO and Shoichi SEINO

(Received on 31 October, 1984)

1. Introduction

Liapunov has discussed the stability of solutions of a system of differential equations by utilizing a scalar function satisfying some conditions.

For the boundedness as well as the stability, the Liapunov's theory is very useful and the relations between Liapunov functions and various types of boundedness are very similar to those between Liapunov functions and various types of stability.

We have also discussed sufficient conditions for the boundedness in [1], [2] and [3].

In this paper, by using the Liapunov's second method, we describe a weak sufficient condition for boundedness and partial boundedness theorem.

2. Definitions and Notations

Let I denote the interval $0 \leqq t < \infty$ and R^n denote Euclidean n−space. Let $|x|$ denote the norm of x for $x \in R^n$. Let $z=(x, y) \in R^n \times R^m$.

We shall denote by $C(I \times R^n \times R^m, R^k)$ the set of all continuous functions defined on $I \times R^n \times R^m$ with values R^k.

Let $F(t, x) \in C(I \times R^n, R^m)$. For a system

$$\frac{dx}{dt}=F(t, x), \qquad \cdots\cdots(1)$$

a solution through a point $(t_0, x_0) \in I \times R^n$ will be denoted by such a form as $x(t, t_0, x_0)$.

Let $f(t, x, y) \in C(I \times R^n \times R^m, R^n)$ and $g(t, x, y) \in C(I \times R^n \times R^m, R^m)$. We consider a system of differential equations

$$\begin{cases} \dfrac{dx}{dt}=f(t, x, y) \\ \dfrac{dy}{dt}=g(t, x, y). \end{cases} \qquad \cdots\cdots(2)$$

Throughout this paper a solution of (2) through a point $(t_0, z_0)=(t_0, x_0, y_0)$ in $I \times R^n \times R^m$ will be denoted by such a form as $(x(t, t_0, z_0), y(t, t_0, z_0))$.

We introduce the following definitions.

[Definition 1] The solutions of (1) are uniformly bounded, if for any $\alpha>0$ and any $t_0 \in I$, there exists a $\beta(\alpha)>0$ such that $|x_0| \leqq \alpha$ implies $|x(t, t_0, x_0)| < \beta(\alpha)$ for all $t \geqq t_0$.

[Definition 2] The solutions of (1) are uniformly ultimately bounded for bound B, if there exists a $B>0$ and if corresponding to any $\alpha>0$ and any $t_0 \in I$, there exists a $T(\alpha)>0$ such that $|x_0| \leqq \alpha$ implies that $|x(t, t_0, x_0)| < B$ for all $t \geqq t_0 + T(\alpha)$.

[Definition 3] The solutions of (2) are partially uniformly bounded with respect to x, if for any $\alpha>0$ and any $t_0 \in I$, there exists a $\beta(\alpha)>0$ such that $|z_0| \leqq \alpha$ implies that

$|x(t, t_0, z_0)| < \beta(\alpha)$ for all $t \geqq t_0$.

[Definition 4] The solutions of (2) are partially uniformly ultimately bounded for bound B with respect to x, if there exists a $B>0$ and if corresponding to any $\alpha>0$ and any $t_0 \in I$, there exists a $T(\alpha)>0$ such that $|z_0| \leqq \alpha$ implies that $|x(t, t_0, z_0)| < B$ for all $t \geqq t_0 + T(\alpha)$.

[Definition 5] Let V(t, x) and V(t, x, y) be continuous scalar functions defined on open sets, and which satisfy locally a Lipschitz condition with respect to x and (x, y) respectively. Corresponding to V(t, x) and V(t, x, y), we define the functions

$$V'_{(1)}(t, x) = \overline{\lim_{h \to +0}} \frac{1}{h} \{V(t+h, x+hF(t, x)) - V(t, x)\}$$

and

$$V'_{(2)}(t, x, y) = \overline{\lim_{h \to +0}} \frac{1}{h} \{V(t+h, x+hf(t, x, y), y+hg(t, x, y)) - V(t, x, y)\}$$

respectively. In case V(t, x) and V(t, x, y) have continuous partial derivatives of the first order, it is evident that

$$V'_{(1)}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot F(t, x)$$

and

$$V'_{(2)}(t, x, y) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t, x, y) + \frac{\partial V}{\partial y} \cdot g(t, x, y),$$

where " · " denotes a scalar product.

3. Preliminary Results

[Theorem 1] *Suppose that there exists a function* $V(t, x) \in C(I \times S_K, R)$, *where* $S_K = \{x \mid |x| \geqq K, x \in R^n\}$ *for a sufficiently large* K, *which satisfies the following conditions;*

(i) $a(t, |x|) \leqq V(t, x) \leqq b(|x|)$, *where the function* a(t, r) *is continuous in* (t, r) *and* $a(t, r) \to \infty$ *uniformly in* t *as* $r \to \infty$, *and the function* b(r) *is continuous,*

(ii) $V'_{(1)}(t, x) \leqq 0$.

Then the solutions of (1) *are uniformly bounded.*

[Theorem 2] *Suppose that there exists a function* $V(t, x, y) \in C(I \times D_K, R)$, *where* $D_K = \{(x, y) \mid |x| + |y| \geqq K, x \in R^n, y \in R^m\}$ *for a sufficiently large* K, *which satisfies the following conditions;*

(i) $a(t, |x|) \leqq V(t, x, y) \leqq b(|x| + |y|)$, *where the function* a(t, r) *is continuous in* (t, r) *and* $a(t, r) \to \infty$ *uniformly in* t *as* $r \to \infty$, *and the function* b(r) *is continuous,*

(ii) $V'_{(2)}(t, x, y) \leqq 0$.

Then the solutions of (2) *are partially uniformly bounded with respect to* x.

4. Main Results

[Theorem 3] *Suppose that there exists a function* $V(t, x) \in C(I \times S_K, R)$, *where* $S_K = \{x \mid |x| \geqq K, x \in R^n\}$ *for a sufficiently large* K, *which satisfies the following conditions;*

(i) $a(t, |x|) \leqq V(t, x) \leqq b(|x|)$, *where the function* a(t, r) *is continuous in* (t, r) *on* $I \times I$, $a(t, r) \to \infty$ *uniformly in* t *as* $r \to \infty$ *and nondeceasing with respect to* t *for each fixed* r *and to* r *for each fixed* t, *and the function* b(r) *is continuous on* I,

(ii) $V'_{(1)}(t, x) \leqq -c(t, |x|)$, *where the function* c(t, r) *is positive, continuous in* (t, r) *on* $I \times I$ *and nondecreasing with respect to* t *for each fixed* r.

Then the solutions of (1) *are uniformly ultimately bounded.*

Proof. By Theorem 1, the solutions of (1) are uniformly bounded, and hence, there exists a $B>0$

such that if $t_0 \in I$, $|x_0| \leqq K$, $|x(t, t_0, x_0)| < B$ for all $t \geqq t_0$. Moreover, for any $\alpha > 0$, there exists a $\beta(\alpha)$ such that if $t_0 \in I$ and $K < |x_0| \leqq \alpha$, $|x(t, t_0, x_0)| < \beta(\alpha)$ for all $t \geqq t_0$. By (ii), there exists a $\gamma(\alpha) > 0$ such that $V'_{(1)}(t, x) \leqq -\gamma$ on $0 \leqq t < \infty$, $K \leqq |x| \leqq \beta(\alpha)$, and by (i) there exists $L(\beta(\alpha)) > 0$ such that $b(|x|) < L(\beta(\alpha))$ on $K \leqq |x| \leqq \beta(\alpha)$.

Let $T = T(\alpha) = \frac{1}{\gamma}\{L(\beta(\alpha)) - a(0, K)\}$. Suppose that $|x(t, t_0, x_0)| > K$ for all $t \in [t_0, t_0 + T]$, we have

$$a(0, K) \leqq a(t, K) \leqq a(t, |x(t, t_0, x_0)|) \leqq V(t, t_0, x_0) \leqq V(t_0, x_0) - \gamma(t - t_0) \leqq b(|x_0|) - \gamma(t - t_0) < L(\beta(\alpha)) - \gamma(t - t_0).$$

If $t = t_0 + T(\alpha)$, we have $a(0, K) < a(0, K)$. This is a contradiction. Thus, there exists a t_1 such that $|x(t_1, t_0, x_0)| \leqq K$ for $t_0 < t_1 \leqq t_0 + T$, and hence, if $t \geqq t_1$ and $K < |x_0| \leqq \alpha$, we have $|x(t, t_0, x_0)| < B$.

Therefore, for any $\alpha > 0$ and any $t_0 \in I$, there exists a $T(\alpha) > 0$ such that $|x_0| \leqq \alpha$ implies $|x(t, t_0, x_0)| < B$ for all $t \geqq t_0 + T(\alpha)$. This shows that the solutions of (1) are uniformly ultimately bounded.

The following theorem can be proved by the same argument as in the proof of Theorem 3.

[Theorem 4] *Suppose that there exists a function* $V(t, x, y) \in C(I \times D_K \times R^m, R)$, *where* $D_K = \{(x, y) \mid |x| + |y| \geqq K, x \in R^n, y \in R^m\}$ *for a sufficiently large* K, *which satisfies the following conditions;*

(i) $a(t, |x|) \leqq V(t, x, y)$, *where the function* $a(t, r)$ *is continuous in* (t, r) *on* $I \times I$, $a(t, r) \to \infty$ *uniformly in* t *as* $r \to \infty$ *and nondecreasing with respect to* t *for each fixed* r *and to* r *for each fixed* t;

(ii) $V(t, x, y) \leqq b(|x| + |y|)$, *where the function* $b(r)$ *is continuous on* I;

(iii) $V'_{(2)}(t, x, y) \leqq -c(t, |x|)$, *where the function* $c(t, r)$ *is positive, continuous in* (t, r) *on* $I \times I$ *and nondecreasing with respect to* t *for each fixed* r.

Then the solutions of (2) *are partially uniformly ultimately bounded.*

References

[1] S. Seino, M. Kudo and M. Aso ; On Stability and Boundedness of Solutions of a System of Ordinary Differential Equations, Research Reports of Akita Technical College, No.9, 1974, 93–98.

[2] S. Seino, M. Kudo and M. Aso ; On the Total Stability and the Total Boundedness of the Perturbed Systems, Reseach Reports of Akita Technical College, No.11, 1976, 100–103.

[3] S. Seino, M. Kudo and M. Aso ; On Partial Boundedness of Solutions of a System of Ordinary Differential Equations, Reseach Reports of Akita Technical College, No.16, 1981, 118–121.

[4] T. Yoshizawa ; Stability Theory by Liapunov's Second Method, Math. Soc. Japan, 1966.

[5] T. Yoshizawa ; Stability Theory and the Existance of Periodic Solutions and Almost Periodic Solutions, Springer–Verlag, 1975.