Riemannian Submersions and Locally Symmetric Submanifolds

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Introduction

In [6] O' Neill introduced the notion of a Riemannian submersion. Let $\pi;M\longrightarrow M'$ be a Riemannian submersion. H. B. Lawson [5] and R. H. Escobales [2] have shown some relations between submanifolds of M and those of M'.

In this paper we will show that if a submanifold N of M is locally symmetric, then a submanifold $\pi(N)$ of M' is also locally symmetric provided some conditions.

1. Submersions

Let M and M' be Riemannian manifolds of dimensions m+p and m respectively. By a Riemannian submersion we mean a C^∞ mapping $\pi;M\longrightarrow M'$ such that π is of maximal rank and π_* preserves the lengths of horizontal vectors, i. e., vectors orthogonal to the fiber $\pi^{-1}(y)$ for some $y\in M'$.

Throughout this paper, we assume that the fibers are totally geodesic in M.

Let X denote a tangent vector at $x\in M$. Then X decomposes as $\mathcal{V}X+\mathcal{H}X$, where $\mathcal{V}X$ is tangent to the fiber through x and $\mathcal{H}X$ is perpendicular to it. If $X=\mathcal{V}X$, it is called a vertical vector ; and if $X=\mathcal{H}X$, it is called horizontal. Let $\tilde{\nabla}$ and $\tilde{\nabla}^*$ denote the Riemannian connections of M and M' respectively.

We define a tensor $\tilde{A}$ associated with the submersion. For any vector fields E and F on M,

$$\tilde{A}_E F=\mathcal{V}\tilde{\nabla}_{\mathcal{H}E}(\mathcal{H}F)+\mathcal{H}\tilde{\nabla}_{\mathcal{H}E}(\mathcal{V}F).$$

$\tilde{A}$ is a (1, 2)-tensor, and it has the following properties [6] :

(1) At each point, $\tilde{A}_E$ is a skew-symmetric linear operator on the tangent space of M, and it reverses the horizontal and vertical subspaces.

(2) $\tilde{A}_E=\tilde{A}_{\mathcal{H}E}$.

(3) For horizontal vector fields, $\tilde{A}$ has the alternation property
$\tilde{A}_X Y=-\tilde{A}_Y X$.

We define a vector field X on M to be basic provided X is horizontal and π-related to a vector field X_* on M'. Every vector field X_* on M' has a unique horizontal lift X to M, and X is basic.

Lemma 1. [6]. If X and Y are basic vector fields on M, then

(1) $\tilde{g}(X, Y)=\tilde{g}^*(X_*, Y_*)\circ\pi$,

(2) $\mathcal{H}[X, Y]$ is the basic vector field corresponding to $[X_*, Y_*]$,

(3) $\mathcal{H}\tilde{\nabla}_X Y$ is the basic vector field corresponding to $\tilde{\nabla}^*_{X_*}Y_*$, where $\tilde{g}$ and $\tilde{g}^*$ are the metrics of M and M' respectively.

Lemma 2. [6]. Let X and Y be horizontal vector fields, and V is vertical vector fields on M. Then

(1) $\tilde{\nabla}_V X=\mathcal{H}\tilde{\nabla}_V X$,

(2) $\tilde{\nabla}_X V=\tilde{A}_X V+\mathcal{V}\tilde{\nabla}_X V$,

(3) $\tilde{\nabla}_X Y=\mathcal{H}\tilde{\nabla}_X Y+\tilde{A}_X Y$.

Furthermore, if X is basic, $H\tilde{\nabla}_V X = \tilde{A}_X V$.

Denote by $\tilde{R}$ the curvature tensor of M. The horizontal lift of the curvature tensor $\tilde{R}^*$ of M' will also be denoted by $\tilde{R}^*$; explicitly, if h_1, h_2, h_3, h_4 are horizontal tangent vectors to M, we set

$$\tilde{g}(\tilde{R}^*_{h_1 h_2}(h_3), h_4) = \tilde{g}^*(\tilde{R}^*_{h_{1*} h_{2*}}(h_{3*}), h_{4*}) \circ \pi$$

where $h_{i*} = \pi_*(h_i)$.

Lemma 3. [6]. Let X, Y, Z, H are horizontal vector fields and V and W are vertical vector fields, then

(1) $\tilde{R}(X, V, Y, W) = \tilde{g}((\tilde{\nabla}_V \tilde{A})_X Y, W) + \tilde{g}(\tilde{A}_X V, \tilde{A}_Y W)$,

(2) $\tilde{R}(X, Y, Z, V) = \tilde{g}((\tilde{\nabla}_Z \tilde{A})_X Y, V)$,

(3) $\tilde{R}(X, Y, Z, H) = \tilde{R}^*(X, Y, Z, H) - 2\tilde{g}(\tilde{A}_X Y, \tilde{A}_Z H) + \tilde{g}(\tilde{A}_Y Z, \tilde{A}_X H) + \tilde{g}(\tilde{A}_Z X, \tilde{A}_Y H)$.

2. Submanifolds

Suppose now that N is an $n+p$−dimensional submanifold of M which respects the submersion π. That is, suppose there is a submersion $\pi ; N \longrightarrow N'$ where N' is a submanifold of M' such that the diagram

$$\begin{array}{ccc} N & \xrightarrow{\quad f \quad} & M \\ \pi \downarrow & f' & \downarrow \pi \\ N' & \longrightarrow & M' \end{array}$$

commutes and the immersion f is a diffeomorphism on the fibers. We assume that the fibers are totally geodesic in N.

Let S be second foundamental form of the submanifold N. Let N_k $(k=1, 2, \cdots\cdots\cdots, m-n)$ be orthonormal normal vector fields of N. Let $g(\nabla)$ and $g^*(\nabla^*)$ denote the induced metrics (connections) of N and N' respectively. Then the Gauss-Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \qquad \tilde{\nabla}_X E = -S_E X + \bar{D}_X E \qquad X, Y \in \mathfrak{X}(N),\ E \in \mathfrak{X}^{\perp}(N),$$

where $g(S_E X, Y) = g(h(X, Y), E)$ and $\bar{D}$ is the connection in the normal bundle $T(N)^{\perp}$. Note that the normal space is always horizontal. We set $C_E X = H S_E H X$ where X is tangent to N. Then we have the following equations

(2. 1) $S_E X = C_E X + \tilde{A}_E X$,

(2. 2) $S_E V = \bar{D}_V E - \tilde{\nabla}_V E = \bar{D}_V E - H\tilde{\nabla}_V E$

where X and V are horizontal and vertical tangent vectors on N[2].

Let A be a tensor associated with the submersion $\pi ; N \longrightarrow N'$. Then we have

$\tilde{A}_X Y = A_X Y + h(HX, VY) \qquad X, Y \in \mathfrak{X}(N)$.

Let $V_i (i=1, 2, \cdots\cdots, p)$ be orthonormal vertical vector fields on N. From $A_X Y$ is vertical, for horizontal vector fields X, Y, we set $A_X Y = \sum_{i=1}^{p} \alpha^i(X, Y) V_i$. Let R and R^* denote the curvature tensors of N and N' respectively. For horizontal vector fields X, Y, Z and H on N we have

(2, 3) $R(X, Y, Z, H) = R^*(X, Y, Z, H) - 2g(A_X Y, A_Z H) + g(A_Y Z, A_X H) + g(A_Z X, A_Y H)$.

We set $D(X, Y, Z, H) = -2g(A_X Y, A_Z H) + g(A_Y Z, A_X H) + g(A_Z X, A_Y H)$.

Lemma 4. Let X, Y, Z, H and C be horizontal vector fields on N. Then

$$\begin{aligned}(\nabla_C D)(X, Y, Z, H) = \sum_{i=1}^{p} \{ & -2\alpha^i(Z, H) R(X, Y, C, V_i) - 2\alpha^i(X, Y) R(Z, H, C, V_i) \\ & + \alpha^i(X, H) R(Y, Z, C, V_i) + \alpha^i(Y, Z) R(X, H, C, V_i) \\ & + \alpha^i(Y, H) R(Z, X, C, V_i) + \alpha^i(Z, X) R(Y, H, C, V_i) \}.\end{aligned}$$

Proof. From the following equations

$$C(D(X,Y,Z,H)) = -2g(\nabla_C(A_XY), A_ZH) - 2g(A_XY, \nabla_C(A_ZH))$$
$$+ g(\nabla_C(A_YZ), A_XH) + g(A_YZ, \nabla_C(A_XH))$$
$$+ g(\nabla_C(A_ZX), A_YH) + g(A_ZX, \nabla_C(A_YH)),$$
$$\nabla_C(A_XY) = (\nabla_C A)_XY + A_{\nabla_C X}Y + A_X(\nabla_C Y)$$

and Lemma 3. (2), we have

$$(\nabla_C D)(X,Y,Z,H) = C(D(X,Y,Z,H)) - D(\nabla_C X, Y, Z, H) - D(X, \nabla_C Y, Z, H)$$
$$- D(X, Y, \nabla_C Z, H) - D(X, Y, Z, \nabla_C H)$$
$$= -2g((\nabla_C A)_XY, A_ZH) - 2g(A_XY, (\nabla_C A)_ZH) + g((\nabla_C A)_YZ, A_XH)$$
$$+ g(A_YZ, (\nabla_C A)_XH) + g((\nabla_C A)_ZX, A_YH) + g(A_ZX, (\nabla_C A)_YH)$$
$$= \sum_{i=1}^{p} \{-2\alpha^i(Z,H)R(X,Y,C,V_i) - 2\alpha^i(X,Y)R(Z,H,C,V_i)$$
$$+ \alpha^i(X,H)R(Y,Z,C,V_i) + \alpha^i(Y,Z)R(X,H,C,V_i)$$
$$+ \alpha^i(Y,H)R(Z,X,C,V_i) + \alpha^i(Z,X)R(Y,H,C,V_i)\}. \qquad \text{q.e.d.}$$

Lemma 5. Let M be a space of constant curvature c. Assum $\tilde{A}_EF=0$, where F is horizontal and tangent to N and E is normal to N. Then $R(X, Y, Z, V)=0$, where X, Y, Z are horizontal vector fields on N and V is vertical vector field on N.

Proof. From the equation of Gauss, we have

$$R(X,Y,Z,V) = \tilde{R}(X,Y,Z,V) - g(h(X,Z), h(Y,V)) + g(h(X,V), h(Y,Z)).$$

By assumption, $\tilde{R}(X, Y)V = c(g(V, Y)X - g(V, X)Y) = 0$. We set $h(X,Y) = \sum_{k=1}^{m-n} h^k(X,Y)N_k$. From Lemma 2, (2. 2) and assumption we obtain

$$g(h(X,Z), h(Y,V)) = \sum_{k=1}^{m-n} h^k(X,Z)g(N_k, h(Y,V)) = \sum_{k=1}^{m-n} h^k(X,Z)g(S_{N_k}V, Y)$$
$$= \sum_{k=1}^{m-n} h^k(X,Z)g(\bar{D}_VN_k - H\tilde{\nabla}_VN_k, Y) = \sum_{k=1}^{m-n} h^k(X,Z)g(-H\tilde{\nabla}_VN_k, Y)$$
$$= \sum_{k=1}^{m-n} h^k(X,Z)g(-\tilde{A}_{N_k}V, Y) = \sum_{k=1}^{m-n} h^k(X,Z)g(V, \tilde{A}_{N_k}Y) = 0. \qquad \text{q.e.d.}$$

Theorem. Let π ; M ⟶ M' be a Riemannian submersion with totally geodesic fibers and N is a submanifold of M which respects the submersion π, that is, there is a submersion π ; N ⟶ N' where N' is a submanifold of M' such that the diagram

$$\begin{array}{ccc} N & \xrightarrow{\ f\ } & M \\ \pi\downarrow & f' & \downarrow\pi \\ N' & \longrightarrow & M' \end{array}$$

commutes and the immersion f is a diffeomorphism on the fibers. We assume that the fibers are totally geodesic in N. Let M be a space of constant curvature and $\tilde{A}_EF=0$, where F is horizontal and tangent to N and E is normal to N. If N is locally symmetric, then N' is also locally symmetric.

Proof. Let X_*, Y_*, Z_*, H_* and C_* be tangent vector fields on N', and let X, Y, Z, H and C be their horizontal lifts. Using (2. 3) and Lemma 1 we see that

$$((\nabla^*_{C_*} R^*)(X_*, Y_*, Z_*, H_*)) \circ \pi$$
$$= (C_*(R^*(X_*, Y_*, Z_*, H_*)) \circ \pi - R^*(\nabla^*_{C_*}X_*, Y_*, Z_*, H_*)) \circ \pi$$
$$- R^*(X_*, \nabla^*_{C_*}Y_*, Z_*, H_*) \circ \pi - R^*(X_*, Y_*, \nabla^*_{C_*}Z_*, H_*) \circ \pi$$
$$- R^*(X_*, Y_*, Z_*, \nabla^*_{C_*}H_*) \circ \pi$$
$$= C(R^*(X,Y,Z,H)) - R^*(H\nabla_C X, Y, Z, H) - R^*(X, H\nabla_C Y, Z, H)$$
$$- R^*(X, Y, H\nabla_C Z, H) - R^*(X, Y, Z, H\nabla_C H)$$
$$= (\nabla_C R)(X,Y,Z,H) + R(V\nabla_C X, Y, Z, H) + R(X, V\nabla_C Y, Z, H)$$
$$+ R(X, Y, V\nabla_C Z, H) + R(X, Y, Z, V\nabla_C H) - (\nabla_C D)(X,Y,Z,H). \qquad \text{q.e.d.}$$

Using Lemma 4, Lemma 5 and assumption we obtain $\nabla^*R^*=0$, in other words, N' is locally symmetric.

Example

Let π ; $S^{2n+1} \longrightarrow CP(n)$ be the standard submersion from a sphere of radius one [2] [6]. Let $\tilde{D}$ be the outward unit normal on the $S^{2n+1} \subset R^{2n+2} = C^{n+1}$. Let J is the natural almost complex structure on C^{n+1}. Let (S^{2n+1}, ϕ, ξ, η, g) be standard Sasakian manifold. A $(2m+1)$−dimensional submanifold N of S^{2n+1} is said to be invariant, if the structure vector field ξ is tangent to N everywhere on N and ϕX is tangent to N for any tangent vector X to N. Any invariant submanifold N with induced structure tensors, which will be denoted by the same letters (ϕ, ξ, η, g) as S^{2n+1}, is also a Sasakian manifold. Let F is horizontal and tangent to N and E is normal to N. Using $\tilde{A}_F(J\tilde{D}) = JF$ and $JF = \phi F + \eta(F)\tilde{D}$ we obtain $g(\tilde{A}_E F, \xi) = -g(\tilde{A}_E F, J\tilde{D}) = g(\tilde{A}_F E, J\tilde{D}) = -g(E, \tilde{A}_F J\tilde{D}) = -g(E, JF) = -g(E, \phi F)$. From N is an invariant submanifold, we see that $g(\tilde{A}_E F, \xi) = 0$. Therefore $\tilde{A}_E F = 0$. The example of a locally symmetric invariant submanifold of S^{2n+1} is an unit sphere S^{2m+1} $(m<n)$ with induced structure. Then $N' = \pi(S^{2m+1}) = CP^m$ is locally symmetric.

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