Symmetries in Lissajous Figures

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1. Introduction

The Hamilton dynamical system with two degrees of freedom under the nonlinear potential energy is one of recent topics in physics and mathematics [1-3]. The Hamiltonian in this system is given by the equation,

$$H = \frac{1}{2}(\omega_1^2 p_x^2 + \omega_2^2 p_y^2) + V(x,y). \tag{1}$$

p_x and p_y are momentums conjugate to coordinates x and y, respectively. ω_1 and ω_2 are frequencies. $V(x,y)$ is the potential energy. The motions in this system can be represented by the phase curves in the four dimensional phase space. The trajectory is the phase curve projected onto the x-y plane. When $V(x,y)$ is nonlinear, this system is in general nonintegrable and the chaotic behaviors can be observed, while the phase curve is on a two dimensional torus in the phase space in the particular cases, in which this system is integrable.

When $V(x,y)$ is equal to the harmonic potential $V_H(x,y) = (x^2+y^2)/2$, the solutions of the Hamilton's canonical equation are

$$x = A_1 \cos(\omega_1 t + \phi_1), \quad y = A_2 \cos(\omega_2 t + \phi_2). \tag{2}$$

A_i and ϕ_i $(i = 1,2)$ are integral constants. The trajectories described by eq. (2) are well known as the Lissajous figures (abbreviated as LF). Putting $\tau = \omega_2 t + \omega_2\phi_1/\omega_1$, we can get the equations,

$$x = A_1 \cos \alpha\tau, \quad y = A_2 \cos(\tau + \phi) \tag{3}$$

with

$$\alpha = \omega_1/\omega_2 \text{ and } \phi = \phi_2 - \phi_1/\alpha. \tag{4}$$

We assume $0 < \alpha < 1$ below. LF can be specified by α and ϕ. LF represent the periodic orbits for rational α although the quasi-periodic orbits for irrational α.

The present author et al. [4] adopted the molecular vibrations in the linear symmetric triatomic molecules, in which the Morse potential is assumed, as an example of the dynamical system with two degrees of freedom and have studied the system. The Morse potential is given by the equation [5],

$$V(x,y) = 2 - 2e^{-x/2}(e^{y/2} + e^{-y/2}) + e^{-x}(e^{y} + e^{-y}). \tag{5}$$

Here, the two body dissociation energy is adopted as the energy unit. The second order term in this potential agrees with the harmonic potential. The third order term is the anti-Hénon-Heiles type [7]. The present author further has analyzed the behaviors of solutions in this classical Morse system in the α-E plane by means of computer experiments (E : total energy) [5]. This analysis has clarified the following facts;

(1) This system is nonintegrable except $\alpha = 1$ and $\alpha = 0$. The chaotic behaviors can be observed macroscopically for $E \gtrsim 0.5$ except the regions near $\alpha = 0$ and $\alpha = 1$ [8].

(2) When α is a irreducible fraction ($\alpha = m/n$), the periodic orbits in this system are survivals of ones in the harmonic potential with the particular phase differences ϕ ($\phi = 0$, π/m for even m, $\phi = 0$,

$\pi/2m$, π/m for odd m). They exist up to the two body dissociation threshold. The periodic orbits with $\phi = 0$ and $\phi = \pi/m$ (m = odd) are stable with respect to the slight change of the initial value and give the n-elliptic fixed points in the Poincaré surfaces of section ($y = 0$, $p_y > 0$). However, those with $\phi = \pi/m$ (m = even) and $\phi = \pi/2m$ (m = odd) are unstable and give the n-hyperbolic fixed points.

(3) The periodic orbit characterized by a rational α becomes to exist in more spreading region including its α with increasing of energy. This leads to more intelligible interpretation of the KAM and Poincaré-Birkhoff fixed point theorems [1-3].

There are some references [6], in which LF for various α and ϕ are listed. But, they are insufficient concerning ϕ. In my study, LF for $\phi = \pi/m$ and $\phi = \pi/2m$ were necessary. The purpose of this note is to make the more complete list of LF and is to clarify the symmetrical properties of LF.

LF for some kinds of rational $\alpha(= m/n)$ and ϕ are drawn in later figures, in which the axis of abscissas denotes the x-axis and that of ordinates the y-axis although they are not shown. In these figures, ϕ is limited to $0 \leqq \phi \leqq \pi/m$. This limitation is due to [Theorem 1] and [Theorem 2] in the next section, while it is sufficient to limit ϕ within $0 \leqq \phi \leqq \pi/2m$ from [Theorem 4].

2. Symmetries in Lissajous Figures

Without loss of generality, we can put $A_1 = A_2 = 1$ in eq. (3). LF specified by α and ϕ is denoted by LF (α, ϕ). Let us assume that α is rational ($\alpha = m/n$, m and n are relatively prime integers).

[Theorem 1] LF $(m/n, \phi)$ is the even function with respect to ϕ.
Proof. The change of variable, $\tau = -t$, is performed in eq. (3). Then, we get the equations,

$$x = \cos \frac{m}{n} t, \quad y = \cos (t - \phi). \tag{5}$$

These represent LF $(m/n, -\phi)$. Therefore, $\mathrm{LF}(m/n, \phi) = \mathrm{LF}(m/n, -\phi)$.

[Theorem 2] $\mathrm{LF}(m/n, \phi)$ is the periodic function with respect to ϕ with a period $2\pi/m$.
Proof. $\mathrm{LF}(m/n, \phi + \phi_p)$ is represented by the equations,

$$x = \cos \frac{m}{n} \tau, \quad y = \cos (\tau + \phi + \phi_p). \tag{6}$$

The change of variable, $\tau = t + \theta$, is carried out. We choose θ so as to satisfy the equations

$$\frac{m}{n} \theta = 2\pi k, \quad \theta + \phi_p = 2\pi \ell \quad (k, \ell : \text{integers}). \tag{7}$$

Then, eq. (6) shows the $\mathrm{LF}(m/n, \phi)$. From eq. (7), we get

$$\phi_p = \frac{2\pi}{m} (m\ell - nk). \tag{8}$$

Since there exist integers k an ℓ satisfying $(m\ell - nk) = 1$ (Appendix), we get $\phi_p = 2\pi/m$. Therefore, LF $(m/n, \phi + 2\pi/m) = \mathrm{LF}(m/n, \phi)$.

[Theorem 3] $\mathrm{LF}(m/n, \phi)$ is symmetric to itself with respect to the x-axis in case (a) (m = even, n = odd), with respect to the y-axis in case (b) (m = odd, n = even) and with respect to the origin in case (c) (m = odd, n = odd), respectively.
Proof. Putting $\tau = t + n\pi$ in eq. (3), we get

$$x = (-1)^m \cos \frac{m}{n} t, \quad y = (-1)^n \cos (t + \phi). \tag{9}$$

In case (a), it is obvious that LF denoted by eq. (9) is symmetric to $\mathrm{LF}(m/n, \phi)$ with respect to the x axis.

The symmetries in other cases are also clear from eq. (9).

[Theorem 4] $LF(m/n,\pi/2m+\phi)$ is symmetric to $LF(m/n,\pi/2m-\phi)$ with respect to the x-axis in cases (b) and (c), with respect to the y-axis in cases (a) and (c) and with respect to the origin in cases (a) and (b), respectively, in which cases (a), (b) and (c) are defined in [Theorem 3].

Proof. $LF(m/n,\pi/2m+\phi)$ are given by the equations,

$$x = \cos\frac{m}{n}\tau, \quad y = \cos(\tau+\pi/2m+\phi). \tag{10}$$

Changing the variable $\tau = -t+\theta$, we get

$$x = \cos(\frac{m}{n}t-\frac{m}{n}\theta), \quad y = \cos(t+\pi/2m-\phi-\theta-\pi/m). \tag{11}$$

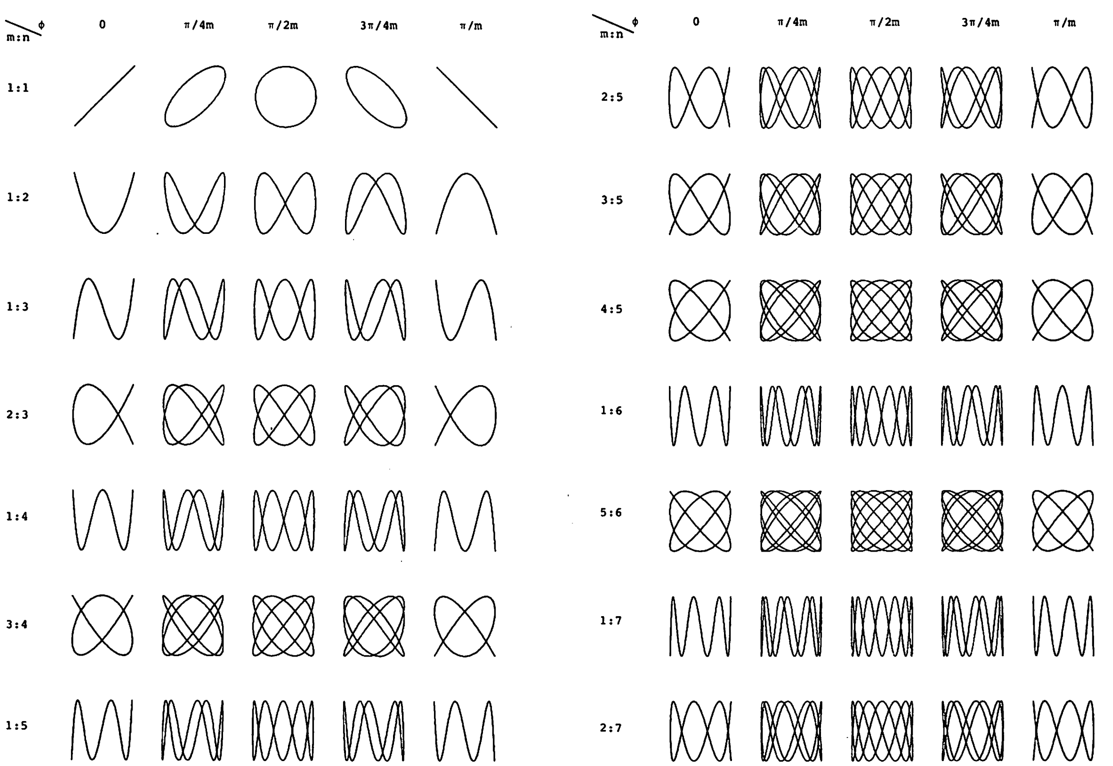
We choose θ so as to satisfy the equations,

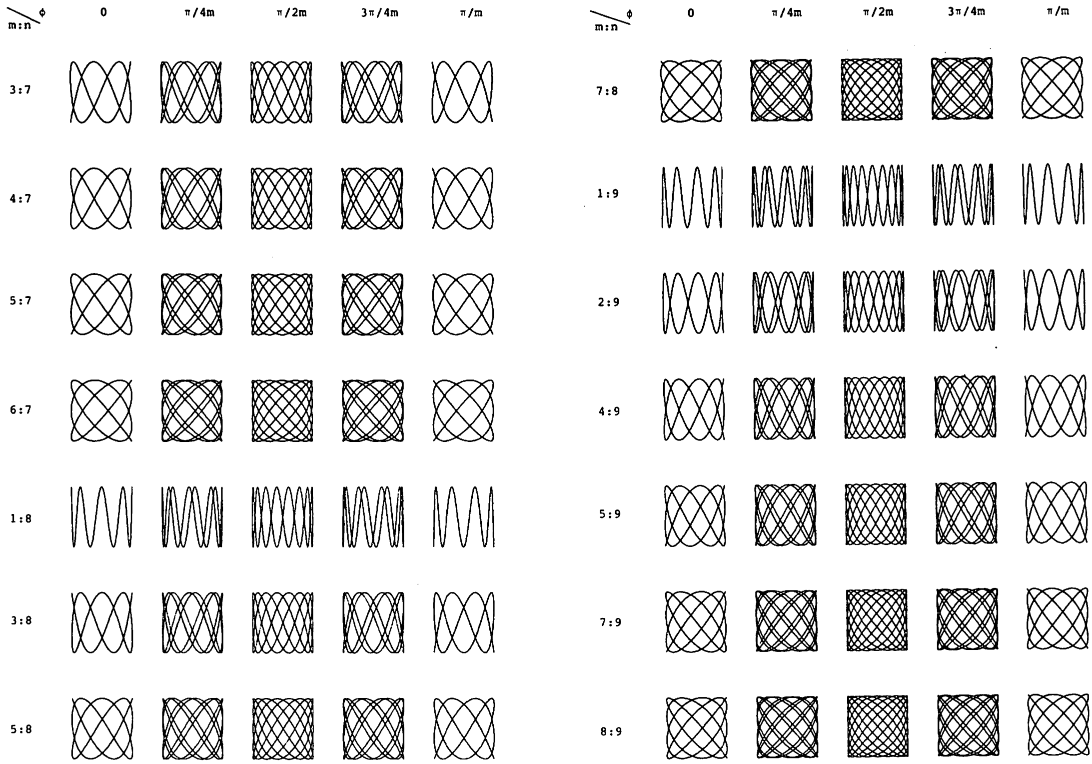
$$\frac{m}{n}\theta = 2\pi k, \quad \theta+\pi/m = (2\ell+1)\pi \quad (k,\ell : \text{integers}). \tag{12}$$

Then, eq. (11) gives LF symmertic to $LF(m/n,\pi/2m-\phi)$ with respect to the x-axis. From eq. (12), we get

$$m(2\ell+1) = 2nk+1. \tag{13}$$

This relation can be satisfied only in cases (b) and (c) (Appendix). In case (a), there are not integers k and ℓ making both hands equal since $m(2\ell+1)$ = even and $2nk+1$ = odd. Other symmetric properties can be also proved in the similar way.





Appendix

[Lemma] If m and n ($|n| > |m|$) are relatively prime integers and s is also an integer, then an infinity of pairs of integers k and ℓ satisfying the following equation exist,

$$m\ell + nk = s. \tag{A}$$

Proof. (i) $m = \pm 1$. In this case, a pair satisfying (A) is $k = 1$ and $\ell = \pm(s-n)$. (ii) $m \mp 1$. (A) is transformed into the equation $m(qk+\ell)+pk = s$ by putting $n = qm+p$ ($|p| < |m|, p, q$: integers). This procedure is repreated until $|p| = 1$. Then, we can find a pair of integers k' and ℓ' satisfying the reduced equation from (i). Next, we can find a pair of integers k and ℓ satisfying (A) by going back the order of the procedure. Therefore, from (i) and (ii), a pair of integers k and ℓ satisfying (A) exists at least.

Let us assume that the pair of k and ℓ is $k = k_0$ and $\ell = \ell_0$. Then, a pair of $k = k_0 - rm$ and $\ell = \ell_0 + rn$, in which r is an integer, satisfies (A). Since an infinity of r exist, an infinity of pairs of k and ℓ satisfying (A) exist.

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