On the Uniform Stability and the Uniformly Asymptotic Stability of a System of Differential Equations

Shoichi SEINO

(Received on 31, July, 1983)

1. Introduction

It is well-known that A. M. Liapunov has discussed the stability of solutions of the system of differential equations by utilizing a scalar function satisfying some conditions. Liapunov's second method is a very useful and powerful instrument in discussing the stability of the system of differential equations.

Its usefulness and power lie in the fact that the criterion of stability can be decided from the differential equations without any knowledge of their solutions.

However, it is great difficult to find the Liapunov function satisfying certain conditions. Therefore, it is important to obtain a weak sufficient condition for the stability theorem. (cf. [1], [2], [3].)

In the previous papers [5], [6], [7], we obtained weak sufficient conditions for the stability and the asymptotic stability.

The purpose of this paper is to give some extension of the sufficient conditions for the uniform stability and the uniformly asymptotic stability.

2. Definitions and Notations

First, we summarize some basic definitions and notations we will need later on.

Let I denote the interval $0 \leqq t < \infty$ and R^n denote Euclidean n-space. For $x \varepsilon R^n$, let $\|x\|$ be any norm of x, and we shall denote by S_H the set of x such that $\|x\| < H$, $H > 0$.

We consider a system of differential equations

(1) $$\frac{dx}{dt} = f(t,x),$$

where x is an n-vector and $f(t,x)$ is an n-vector functions.

Suppose that $f(t,x)$ is continuous on $I \times S_H$ and that $f(t,x)$ is smooth enough to ensure existence, uniqueness and continuous dependence of the solutions of the initial value problem associated with (1), moreover, $f(t,0) \equiv 0$.

We shall denote by $C(I \times R^n, R^n)$ the set of all continuous functions defined on $I \times R^n$ with valued in R^n. Throughout this paper a solution through a point $(t_0, x_0) \varepsilon I \times R^n$ will be denoted by such a form as $x(t, t_0, x_0)$.

We introduce the following definitions.

[Definition 1.] The zero solution of the system (1) is said to be stable if for any $\varepsilon > 0$ and any $t_0 \varepsilon I$ there exists a $\delta(\varepsilon, t_0) > 0$ such that the inequality $\|x_0\| < \delta(\varepsilon, t_0)$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geqq t_0$.

[Definition 2.] The zero solution of the system (1) is said to be uniformly stable if δ of [Definition 1.] is independent of t_0.

[Definition 3.] The zero solution of the system (1) is said to be uniformly attractive if for any $\varepsilon > 0$ and any $t_0 \varepsilon I$ there exist $\delta_0 > 0$ and $T(\varepsilon) > 0$ such that the inequality $\|x_0\| < \delta_0$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geqq t_0 + T(\varepsilon)$.

〔Definition 4.〕 The zero solution of the system (1) is said to be uniformly asymptotically stable if it is uniformly stable and uniformly attractive.

〔Definition 5.〕 Corresponding to a continuous scalar function $V(t,x)$ defined on an open set, we define the function

$$\dot{V}_{(1)}(t,x) = \overline{\lim_{h\to 0+}}\frac{1}{h}\{V(t+h,x+hf(t,x))-V(t,x)\}.$$

In case $V(t,x)$ has continuous partial derivatives of the first order, it is evident that

$$\dot{V}_{(1)}(t,x) = \frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}\cdot f(t,x),$$

where "・" denote a scalar product.

3. Preliminary Results

A. M. Liapunov and T. Yoshizawa gave sufficient conditions for the stability, the uniform stability and the uniformly asymptotic stability. Here, we repeat those theorems.

〔Theorem 3.1.〕 *Suppose that there exists a Liapunov function* $V(t,x)$ *defined on* $I\times S_H$ *which satisfies the following conditions :*

(i) $V(t,0)\equiv 0$,

(ii) $a(\|x\|)\leqq V(t,x)$, *where* $a(r)$ *is continuous, increasing and positively definite,*

(iii) $\dot{V}_{(1)}(t,x)\leqq 0$,

Then, the solution $x=0$ *of the system* (1) *is stable.*

〔Theorem 3.2.〕 *If condition* (ii) *in Theorem 3.1. is replaced by*

(ii)' $a(\|x\|)\leqq V(t,x)\leqq b(\|x\|)$, *where* $a(r)$ *and* $b(r)$ *are continuous, increasing and positively definite,*

the solution $x=0$ *of the system* (1) *is uniformly stable.*

〔Theorem 3.3.〕 *Under the same assumptions as in Theorem 3.2., if* $\dot{V}_{(1)}(t,x)\leqq -c(\|x\|)$, *where* $c(r)$ *is continuous on* $[0,H]$ *and is positively definite, then the solution* $x=0$ *of the system* (1) *is uniformly asymptotically stable.*

〔Theorem 3.4.〕 *Suppose that there exists a Liapunov function* $V(t,x)$ *defined on* $I\times S_H$, *which satisfies the following conditions :*

(i) $a(\|x\|)\leqq V(t,x)\leqq b(\|x\|)$, *where* $a(r)$ *and* $b(r)$ *are continuous, increasing and positively definite,*

(ii) $\dot{V}_{(1)}(t,x)+V^*(t,x)\to 0$ *uniformly on* $0<\gamma\leqq\|x\|<H$, *for any* $\gamma>0$, *as* $t\to\infty$, *where* $V^*(t,x)$ *is continuous and there exists a continuous function* $c(r)>0$ *for* $0<r<H$ *such that* $c(\|x\|)\leqq V^*(t,x)$.

Then, if the solution $x=0$ *of the system* (1) *is unique to the right, it is uniformly asymptotically stable.*

For the proof of these theorems, see [2], [3].

4. Main Results

Before we state main results, we give the following lemma we shall need later on.

〔Lemma〕 *Suppose that* $f(t,x)$ *of the system* (1) *is continuous on* $a\leqq t\leqq b$, $\|x-x_0(t)\|\leqq r$, $r>0$, *where* $x_0(t)$ *is a solution defined on* $a\leqq t\leqq b$, *and that* $x_0(t)$ *is unique to the right. Then, given any* $\varepsilon>0$, *there exists a* $\delta>0$ *such that if* $a\leqq t_0\leqq b$ *and* $\|x_0-x_0(t_0)\|<\delta$, *we have* $\|x(t,t_0,x_0)-x_0(t)\|<\varepsilon$ *for* $t\,\varepsilon\,[t_0,b]$.

For the proof, see [2].

[Theorem 4.1.] *Suppose that there exists a function* $V(t,x) \varepsilon C(I \times S_H, R)$ *which satisfies the following conditions :*

(i) $a(t,\|x\|) \leqq V(t,x)$, *where a function* $a(t,r)$ *is continuous in* (t,r), $a(t,r) > 0$ *for any* $r \neq 0$ *and* $a(t,0) \equiv 0$,

(ii) $V(t,0) \equiv 0$

(iii) $\dot{V}_{(1)}(t,x) \leqq 0$.

Then the zero solution $x = 0$ *of the system* (1) *is stable.*

Proof. For any $\varepsilon > 0$ $(0 < \varepsilon < H)$, if we assume that $\inf\{V(t,x) : t \varepsilon I, \|x\| \geqq \varepsilon\} = 0$, then there exists $(t_1,x_1) \varepsilon I \times S_H$ such that $\|x_1\| \geqq \varepsilon$ and $V(t_1,x_1) = 0$. On the other hand, for $t_1 \varepsilon I$, $\|x_1\| \geqq \varepsilon$, we obtain $V(t_1,x_1) \geqq a(t_1, \|x_1\|) > 0$ by (i). This is a contradiction. Therefore we have $0 < \inf\{V(t,x) : t \varepsilon I, \|x\| \geqq \varepsilon\}$. Hence, if we take a positive number $\rho(\varepsilon) > 0$ such that $\rho(\varepsilon) < \inf\{V(t,x) : t \varepsilon I, \|x\| \geqq \varepsilon\}$, then we have $\|x\| < \varepsilon$ when $t \geqq 0$ and $V(t,x) < \rho(\varepsilon)$.

Since $V(t,x)$ is continuous on $I \times S_H$, for any $t_0 \varepsilon I$ and $\rho(\varepsilon) > 0$ there is a $\delta = \delta(\varepsilon, t_0) > 0$ such that $V(t_0,x_0) < \rho(\varepsilon)$ for $\|x_0\| < \delta$.

We get $V(t,x(t,t_0,x_0)) \leqq V(t_0,x_0)$, because, by (iii), the function $V(t,x(t,t_0,x_0))$ is nonincreasing in t.

Hence, for any $\varepsilon > 0$ and any $t_0 \varepsilon I$, there is a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\|x_0\| < \delta$ implies $V(t,x(t,t_0,x_0)) \leqq V(t_0,x_0) < \rho(\varepsilon)$ for all $t \geqq t_0$.

Therefore $\|x_0\| < \delta$ implies $\|x(t,t_0,x_0)\| < \varepsilon$ for all $t \geqq t_0$, which shows the stability of the zero solution of (1).

[Theorem 4.2.] *Suppose that there exists a function* $V(t,x) \varepsilon C(I \times S_H, R)$, *which satisfies the following conditions :*

(i) $a(t,\|x\|) \leqq V(t,x)$, *where a function* $a(t,r)$ *is continuous in* (t,r), $a(t,r) > 0$ *for any* $r \neq 0$, *and* $a(t,0) \equiv 0$,

(ii) $V(t,x) \leqq b(\|x\|)$, *where a function* $b(r)$ *is continuous, increasing and positively definite,*

(iii) $\dot{V}_{(1)}(t,x) \leqq 0$.

Then the zero solution $x = 0$ *of the system* (1) *is uniformly stable.*

Proof. By the same argument as in Theorem 4.1., for any $\varepsilon > 0$, we can choose $\rho(\varepsilon) > 0$ such that $\rho(\varepsilon) < \inf\{V(t,x) : t \varepsilon I, \|x\| \geqq \varepsilon\}$ and hence we have $\|x\| < \varepsilon$ when all $t \varepsilon I$ and $V(t,x) < \rho(\varepsilon)$. From (ii), we can choose a $\delta = \delta(\varepsilon) > 0$ such that $b(\delta) < \rho(\varepsilon)$.

By (iii), as the function $V(t,x(t,t_0,x_0))$ is nonincreasing in t, $V(t,x(t,t_0,x_0)) \leqq V(t_0,x_0)$.

Therefore, for any $\varepsilon > 0$ and any $t_0 \varepsilon I$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|x_0\| < \delta$ implies $V(t,x(t,t_0,x_0)) \leqq V(t_0,x_0) \leqq b(\delta) < \rho(\varepsilon)$ for all $t \geqq t_0$, hence $\|x(t,t_0,x_0)\| < \varepsilon$ for all $t \geqq t_0$. This proves the uniform stability of the zero solution of (1).

Next, we shall discuss the theorems on the uniformly asymptotic stability.

[Theorem 4.3.] *Suppose that there exists a function* $V(t,x) \varepsilon C(I \times S_H, R)$, *which satisfies the following conditions :*

(i) $a(t, \|x\|) \leqq V(t, x)$, *where a function* $a(t, r)$ *is continuous in* (t, r), $a(t, r) >$ *for any* $r \neq 0$, $a(t, 0) \equiv 0$, *increases with respect to t for each fixed r and increases with respect to r for each fixed t,*

(ii) $V(t,x) \leqq b(\|x\|)$, *where a function* $b(r)$ *is continuous, increasing and positively definite,*

(iii) $\dot{V}_{(1)}(t,x) \leqq -c(t,\|x\|)$, *where a function* $c(r)$ *is continuous in* (t,r) $c(t,r) > 0$ *for any* $r \neq 0$ *and* $c(t,0) \equiv 0$.

Then the zero solution $x = 0$ *of the system* (1) *is uniformly asymptotically stable.*

Proof. By Theorem 4.2., the zero solution of the system (1) is uniformly stable.

We shall prove that the zero solution of the system (1) is uniformly attractive.

By the uniform stability, for any $t_0 \varepsilon I$ and $H > 0$, there exists a $\delta_0 = \delta_0(H) > 0$ for which

(2) $\|x(t,t_0,x_0)\| \leqq H$ for all $t \geqq t_0$ if $\|x_0\| < \delta_0$,

and for any $t_0 \varepsilon I$ and any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $\|x_0\| < \delta$ implies

(3) $\|x(t,t_0,x_0)\| < \varepsilon$ for all $t \geqq t_0$.

Consider a solution $x(t,t_0,x_0)$ of (1) such that $t_0 \varepsilon I$ and $\|x_0\| < \delta_0$. Then there is a $t^* > t_0$ such that $\|x(t^*,t_0,x_0)\| < \delta$.

In fact, if we assume that this is not true, then $\delta \leqq \|x(t,t_0,x_0)\| \leqq H$ for all $t \geqq t_0$. We put $\gamma = \inf\{c(t,r) : t \varepsilon I, \delta \leqq r \leqq H\}$, then $0 < \gamma \leqq c(t,\|x\|)$ for $\delta \leqq \|x\| \leqq H$. From (iii), for all $t \geqq t_0 + T(\varepsilon)$, where $T(\varepsilon) = \frac{1}{\gamma}\{b(\delta_0) - a(0,\delta)\}$, $V(t,x(t,t_0,x_0)) \leqq V(t_0,x_0) - \gamma(t-t_0) \leqq b(\|x_0\|) - \gamma T(\varepsilon) < a(0,\delta)$,

on the other hand, by (i),

$a(0,\delta) \leqq a(t,\|x(t,t_0,x_0)\|) \leqq V(t,x(t,t_0,x_0))$, which contradicts the assumption we have made. Thus, by (3), we must have $\|x(t,t_0,x_0)\| < \varepsilon$ for all $t \geqq t^*$.

Therefore, $\|x(t,t_0,x_0)\| < \varepsilon$ for all $t \geqq t_0 + T(\varepsilon)$, where clearly $T(\varepsilon)$ dependents only on ε. This shows that the zero solution of (1) is uniformly attractive. Thus, the proof is completed.

〔Theorem 4.4〕 *Suppose that there exist two functions* $V(t,x) \varepsilon C(I \times S_H, R)$, $W(t,x) \varepsilon C(I \times S_H, R)$ *satisfying the following conditions :*

(i) $a(t,\|x\|) \leqq V(t,x)$, *where a function* $a(t,r)$ *is continuous in* (t,r), $a(t,r) > 0$ *for any* $r \neq 0$, $a(t,0) \equiv 0$, *increases with respect to t for each fixed r and increases with respect to r for each fixed t,*

(ii) $V(t,x) \leqq b(\|x\|)$, *where a function* $b(r)$ *is continuous, increasing and positively definite,*

(iii) $c(t,\|x\|) \leqq W(t,x)$, *where a function* $c(t,r)$ *is continuous in* (t,r), $c(t,r) > 0$ *for any* $r \neq 0$ *and* $c(t,0) \equiv 0$,

(iv) *for any* $\rho_1 > 0$ *and* $\rho_2 > 0$,
$\dot{V}_{(1)}(t,x) + W(t,x) \longrightarrow 0$ *as* $t \longrightarrow \infty$, *uniformly with respect to x such that* $\rho_1 \leqq \|x\| \leqq \rho_2$.

Then the zero solution $x = 0$ *of the system* (1) *is uniformly asymptotically stable.*

Proof. First, let us prove the uniform stability of the zero solution of (1).

By the properties of the function $b(r)$, for any $\varepsilon > 0$ $(0 < \varepsilon \leqq H^* < H)$, we can choose $\delta = \delta(\varepsilon) > 0$ so small that $b(\delta) < a(0,\varepsilon)$, $\delta < \varepsilon$.

We put $D(\delta,H^*) = \{x : \delta \leqq \|x\| \leqq H^*\}$. On the domain $I \times D(\delta,H^*)$, there exist a $\gamma > 0$ and a $T(\varepsilon) > 0$ such that $\gamma \leqq W(t,x)$ and if $t \geqq T(\varepsilon)$, by (iv), we have $\dot{V}_{(1)}(t,x) + W(t,x) \leqq \frac{1}{2}\gamma$, hence

(4) $\dot{V}_{(1)}(t,x) \leqq -\frac{1}{2}\gamma$ for all $t \geqq T(\varepsilon)$, all $x \varepsilon D(\delta,H^*)$.

Consider a solution $x(t,t_0,x_0)$ of (1) such that $t_0 \geqq T(\varepsilon)$, $\|x_0\| < \delta$.

We shall prove that $\|x(t,t_0,x_0)\| < \varepsilon$ for all $t \geqq t_0$, $\|x_0\| < \delta$.

If we assume that this is not true, then there are $t_1, t_2, t_0 < t_1 < t_2$, such that $\|x(t_1,t_0,x_0)\| = \delta$, $\|x(t_2,t_0,x_0)\| = \varepsilon$ and that $\delta < \|x(t,t_0,x_0)\| < \varepsilon$ for all $t \varepsilon (t_1,t_2)$. By virtue of (i) and (4), it follows that

$a(0,\varepsilon) < a(t_2,x(t_2,t_0,x_0)) \leqq V(t_2,x(t_2,t_0,x_0)) \leqq V(t_1,x(t_1,t_0,x_0)) \leqq b(\delta) < a(0,\varepsilon)$, which contradicts. Therefore, we have $\|x(t,t_0,x_0)\| < \varepsilon$ for all $t \geqq t_0$, $\|x_0\| < \delta$.

From the lemma, there exists a $\delta_1 = \delta_1(\varepsilon) > 0$ such that if $0 \leqq t_0 < T(\varepsilon)$ and $\|x_0\| < \delta_1$, then we have $\|x(t,t_0,x_0)\| < \delta$ for all $t \in [t_0, T(\varepsilon)]$, in particular $\|x(T(\varepsilon),t_0,x_0)\| < \delta$. Thus, if $t_0 \varepsilon I$ and $\|x_0\| < \delta_1$, $\|x(t,t_0,x_0)\| < \varepsilon$ for all $t \geqq t_0$, which proves the uniform stability of zero solution of (1).

Next we shall show that the zero solution of (1) is uniformly attractive.

From the uniform stability, for any $t_0 \varepsilon I$ and $H^* > 0$, there is a $\delta_0 = \delta_0(H^*) > 0$ such that $\|x_0\| < \delta_0$ implies

(5) $\|x(t,t_0,x_0)\| \leqq H^*$ for all $t \geqq t_0$,

and for any $\varepsilon > 0$ and any $t_0 \varepsilon I$, there is $\delta = \delta(\varepsilon) > 0$ for which

(6) $\|x(t,t_0,x_0)\| < \varepsilon$ for all $t \geqq t_0$, if $\|x_0\| < \delta$.

In the case $t_0 \geqq T(\varepsilon)$, where $T(\varepsilon)$ is that obtained in the uniform stability proof above, we consider a solution $x(t,t_0,x_0)$ of (1) such that $t_0 \geqq T(\varepsilon)$ and $\|x_0\| < \delta_0$. For this solution , there exists $t^* > t_0$ such that $\|x(t^*,t_0,x_0)\| < \delta$.

Let us assume that this is not true, then $\delta \leqq \|x(t,t_0,x_0)\| \leqq H^*$ for all $t \geqq t_0$.

On the domain $t \geqq T(\varepsilon)$, $x \varepsilon D(\delta,H^*)$, we have (4) and hence, it can be seen that $V(t,x(t,t_0,x_0)) < a(0,\delta)$ if $t \geqq t_0 + T_1(\varepsilon)$, where $T_1(\varepsilon) = \frac{2}{\gamma}\{b(\delta_0) - a(0,\delta)\}$.

On the other hand, from (i), $a(0,\delta) \leqq V(t,x(t,t_0,x_0))$, which contradicts the assumption. Therefore, according to (6), we have $\|x(t,t_0,x_0)\| < \varepsilon$ for all $t \geqq t^*$.

Thus, $\|x(t,t_0,x_0)\| < \varepsilon$ for all $t \geqq t_0 + T_1(\varepsilon)$.

Consider now the case $0 \leqq t_0 < T(\varepsilon)$. For the solution $x(t,t_0,x_0)$ such that $0 \leqq t_0 < T(\varepsilon)$, to prove the uniform attractivity of the zero solution of (1) it is sufficient to show that there exists a $\delta_1 = \delta_1(\varepsilon) > 0$ such that $\|x(T(\varepsilon),t_0,x_0)\| < \delta$. For this purpose, applying the lemma again, we can found a $\delta_1 = \delta_1(\varepsilon) > 0$ for which $\|x_0\| < \delta_1$ implies $\|x(t,t_0,x_0)\| < \delta$ for all $t \in [t_0, T(\varepsilon)]$, and in particular, $\|x(T(\varepsilon),t_0,x_0)\| < \delta$, which shows that the zero solution of (1) is uniformly attractive. This completes the proof.

References

〔1〕 A. M. Liapunov: Problème général de la stabilité du mouvement, Annals of Math. Studies 17, Princeton Univ. Press, 1949.

〔2〕 T. Yoshizawa: Stability theory by Liapunov's second method, Math. Soc. Japan, 1966.

〔3〕 T. Yoshizawa: Stability theory and the existence of periodic solutions and almost periodic solutions, Springer, 1975.

〔4〕 V. M. Matrosov: On the theory of stability of motions, J. Appl. Math. Mech., 26, 1962, 1506-1522.

〔5〕 S. Seino, M. Kudo and M. Aso: Some generalization of Liapunov's stability theorem, Research Reports of Akita Technical College, 8, 1973, 126-131.

〔6〕 S. Seino, M. Kudo and M. Aso: On stability and boundedness of solutions of a system of ordinary differential equations, Research Reports of Akita Technical College, 9, 1974, 93-98.

〔7〕 S. Seino, M. Kudo and M. Aso: Some result on asymptotic stability of solutions of ordinary differential equations, Research Reports of Akita Technical College, 14, 1979, 89-92.