Partial Stability Theorems by Liapunov's Second Method

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1. Introduction

Liapunov's second method enables us to decide stability and boundedness from the differential equations without any knowledge of their solutions.

However, it is difficult to find Liapunov's function $V(t,x)$ satisfying certain conditions. Therefore, it is important to obtain a weak sufficient condition for stability and boundedness theorem.

In this paper, by using this method, we give sufficient conditions for stability and asymptotic stability relative to a part of the variables.

2. Definitions and Notations

Let I denote the interval $0 \leqq t < \infty$ and R^n denote Euclidean n-space.

We consider the system

$$\frac{dx}{dt} = f(t,x), \qquad (1)$$

where the function $f(t,x)$ is defined and is continuous in (t,x) on $I \times R^n$, and $f(t,0)=0$.

We shall study the question of the stability of the solution $x=0$ relative to $x_1, \cdots\cdots, x_m, (0<m\leqq n)$. Denoting these variables by $y_i = x_i$ $(i=1,\cdots\cdots,m)$, we introduce the notation $|y| = (\sum_{i=1}^{m} y_i^2)^{\frac{1}{2}}$, $|x| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$.

We assume that,

(a) in the region $t \geqq 0$, $|y| \leqq H$ $(H>0)$, $f(t,x)$ satisfies the conditions for the uniqueness of the solution,

(b) any solution $x(t)$ is defined for all $t \geqq 0$ for which $|y(t)| \leqq H$ $(H>0)$.

By $x=x(t,t_0,x_0)$, we denote the solution of (1) satisfying the initial conditions $x(t_0,t_0,x_0)=x_0$.

We introduce the following definitions.

【Definition 1】 The solution $x=0$ is y−stable if for any $\varepsilon>0$ and any $t_0 \in I$, there exists a $\delta(t_0,\varepsilon)>0$ such that $|x_0|<\delta$ implies $|y(t,t_0,x_0)|<\varepsilon$ for all $t \geqq t_0$.

【Definition 2】 The solution $x=0$ is uniformly y−stable if the δ above is independent of t_0.

【Definition 3】 The solution $x=0$ is asymptotically $y-$stable if it is $y-$stable and if there exists a $\delta_0(t_0)>0$ such that $|x_0|<\delta_0(t_0)$ implies $y(t,t_0,x_0)\longrightarrow 0$ as $t\longrightarrow\infty$.

【Definition 4】 The solution $x=0$ is asymptotically $y-$stable uniformly in x_0 from the region D: $\sum_{i=1}^{k} x_{i0}^2<\delta^2$, $-\infty<x_{j0}<\infty$ $(j=k+1,\cdots\cdots,n)$ for $\delta>0$ if it is uniformly $y-$stable and if given any $\varepsilon>0$ and any $t_0\in I$, there exist $\delta>0$ and $T(t_0,\varepsilon)>0$ such that if $x_0\in D$, $|y(t,t_0,x_0)|<\varepsilon$ for all $t\geqq t_0+T$.

【Definition 5】 For a continuous function $V: I\times R^n\rightarrow I$, we define the function

$$V'_{(1)}(t,x)=\lim_{h\to+0}\frac{1}{h}\{V(t+h,x+hf(t,x))-V(t,x)\}.$$

3. Preliminary Results

【Theorem 1】 *If there exists a continuous function* $V: I\times R^n\rightarrow I$ *such that*

(i) $V(t,0)=0$,

(ii) $a(t,|y|)\leqq V(t,x)$, *where the continuous function* $a: I\times I\rightarrow I$ *is* $a(t,0)=0$, $a(t,r)>0$ *for* $r\neq 0$ *and increases monotonically with respect to* t *for each fixed* r,

(iii) $V'_{(1)}(t,x)\leqq 0$,

then the solution $x=0$ *is* $y-$*stable*.

For the proof, see [1].

【Theorem 2】 *If there exists a continuous function* $V: I\times R^n\rightarrow I$ *such that*,

(i) $a(|y|)\leqq V(t,x)$, *where* $a(r)$ *is continuous increasing, positive definite function*,

(ii) $V'_{(1)}(t,x)\leqq 0$ *on* $I\times R^n$,

(iii) $V'_{(1)}(t,x)\leqq -m_\eta(t)$ *for* t *and* x *such that for any* $\eta>0$ $V(t,x)\geqq\eta$ *and* $|y|\leqq H$ $(H>0)$, *where* $\int_0^\infty m_\eta(t)\,dt=\infty$,

then the solution $x=0$ *is asymptotically* $y-$*stable*.

For the proof, see [2].

【Theorem 3】 *If there exists a continuous function* $V: I\times R^n\rightarrow I$ *such that*,

(i) $a(|y|)\leqq V(t,x)\leqq b((\sum_{i=1}^{k} x_i^2)^{\frac{1}{2}})$ *for* $m\leqq k\leqq n$, *where* $a(r)$ *and* $b(r)$ *are continuous increasing, positive definite function*,

(ii) $V'_{(1)}(t,x)\leqq 0$ *on* $I\times R^n$,

(iii) $V'_{(1)}(t,x)\leqq -m_\eta(t)$ *for* t *and* x *such that* $\sum_{i=1}^{k}x_i^2\geqq\eta^2$ *for any* $\eta>0$ *and* $|y|\leqq H$, $(H>0)$, *where* $\int_0^\infty m_\eta(t)\,dt=\infty$,

then the solution $x=0$ *is asymptotically* $y-$*stable uniformly in* x_0 *from the region* D.

For the proof, see [2].

4. Main Results

【Theorem 4】 *If there exists a continuous function* $V : I \times R^n \to I$ *such that,*

(i) $a(t, |y|) \leqq V(t,x)$, *where the continuous function* $a : I \times I \to I$ *is* $a(t, 0) = 0$, $a(t,r) > 0$ *for* $r \neq 0$ *and increases monotonically with respect to* t *and* r.

(ii) $V'_{(1)}(t,x) \leqq 0$ *on* $I \times R^n$,

(iii) $V'_{(1)}(t,x) \leqq -m_\eta(t)$ *for* t *and* x *such that for any* $\eta > 0$, $V(t,x) \geqq \eta$ *and* $|y| \leqq H\ (H > 0)$, *where* $\int_0^\infty m_\eta(t)\, dt = \infty$,

then the solution $x = 0$ *is asymptotically* y*-stable.*

Proof. By [1], the solution $x=0$ is y-stable. Let us show that $|x_0| < \delta_0$ implies $V(t, x(t, t_0, x_0)) \to 0$ as $t \to \infty$, where the δ_0 is the same number as in y-stable. Suppose that it is not so. Because $V'(t, x(t, t_0, x_0)) \leqq 0$, for some $\eta > 0$ we have $V(t, x(t, t_0, x_0)) \geqq \eta$. Since $|y(t, t_0, x_0)| < \varepsilon < H$, from (iii), it follows that

$$V(t, x(t, t_0, x_0)) = V(t_0, x_0) + \int_{t_0}^{t} V'(\tau, x(\tau, t_0, x_0))\, d\tau \leqq V(t_0, x_0) - \int_{t_0}^{t} m_\eta(\tau)\, d\tau \to -\infty$$

as $t \to \infty$. Therefore, we have $V(t, x(t, t_0, x_0)) \to 0$ as $t \to \infty$, and hence, $|y(t, t_0, x_0)| \to 0$ as $t \to \infty$ from (i) and (ii). Thus, the solution $x = 0$ is asymptotically y-stable.

【Theorem 5】 *If there exists a continuous function* $V : I \times R^n \to I$ *such that,*

(i) $a(t, |y|) \leqq V(t,x) \leqq b\left(\left(\sum_{i=1}^{k} x_i^2\right)^{\frac{1}{2}}\right)$ *for* $m \leqq k \leqq n$, *where the continuous function* $a : I \times I \to I$ *is* $a(t, 0) = 0$, $a(t,r) > 0$ *for* $r \neq 0$ *and increases monotonically with respect to* t *for each fixed* r, *and* $b(r)$ *is continuous increasing, positive definite function,*

(ii) $V'_{(1)}(t, x) \leqq 0$ *on* $I \times R^n$,

(iii) $V'_{(1)}(t,x) \leqq -m_\eta(t)$ *for* t *and* x *such that* $\sum_{i=1}^{k} x_i^2 \geqq \eta^2$ *for any* $\eta > 0$ *and* $|y| \leqq H\ (H > 0)$, *where* $\int_0^\infty m_\eta(t)dt = \infty$,

then the solution $x = 0$ *is asymptotically* y*-stable uniformly in* x_0 *from the region* D.

Proof. First of all, we shall see that the solution $x=0$ is uniformly y-stable. For any $\varepsilon > 0$, we can choose a $\delta(\varepsilon) > 0$ so that $b(\delta) < a(0, \varepsilon)$.

We assume that a solution $y(t, t_0, x_0)$ for $x_0 \in D$ satisfies $|y(t_1, t_0, x_0)| = \varepsilon$ at some $t_1 > t_0$. From (i) and (ii), it follows that

$$a(0, \varepsilon) < a(t_1, \varepsilon) = a(t_1, |y(t_1, t_0, x_0)|) \leqq V(t_1, x(t_1, t_0, x_0)) < V(t_0, x_0)$$
$$\leqq b\left(\left(\sum_{i=1}^{k} x_{i0}^2\right)^{\frac{1}{2}}\right) < b(\delta) < a(0, \varepsilon).$$

This is a contradiction, and hence, if $x_0 \in D$, then $|y(t, t_0, x_0)| < \varepsilon$ for all $t \geqq t_0$, which shows that the solution $x=0$ is uniformly y-stable.

By (iii), for any $\varepsilon > 0$ and any $t_0 \in I$ there exist $\delta_0 > 0$ and $T(t_0, \varepsilon) > 0$

such that $b(\delta_0) = a(0, H)$ and $\int_{t_0}^{t_0+T} m_\eta(\tau)\, d\tau = a(0, H)$ for $b(\eta) = a(0, \varepsilon)$.

We assume that $V(t, x(t, t_0, x_0)) \geqq a(0, \varepsilon)$ for all $t \in (t_0, t_0+T]$. From (i) and (ii), it follows that

$$0 < a(0, \varepsilon) \leqq V(t_0+T, x(t_0+T, t_0, x_0)) = V(t_0, x_0) + \int_{t_0}^{t_0+T} V'(\tau, x(\tau, t_0, x_0))\, d\tau .$$

Since the solution $x = 0$ is uniformly y-stable, $|y(t, t_0, x_0)| \leqq H$ for all $t \geqq t_0$ and further, by (i), it follows that

$$b(\eta) = a(0, \varepsilon) \leqq V(t, x(t, t_0, x_0)) \leqq b((\sum_{i=1}^{k} x_i^2(t, t_0, x_0))^{\frac{1}{2}}),$$

and hence, $\eta^2 \leqq \sum_{i=1}^{k} x_i^2(t, t_0, x_0)$. Thus, the conditions of (iii) is satisfied, therefore, we have

$$\int_{t_0}^{t_0+T} V'(\tau, x(\tau, t_0, x_0))\, d\tau \leqq -\int_{t_0}^{t_0+T} m_\eta(\tau)\, d\tau = -a(0, H),$$

and it follows that

$$0 < a(0, \varepsilon) \leqq V(t_0, x_0) - a(0, H) \leqq b((\sum_{i=1}^{k} x_{i0}^2)^{\frac{1}{2}}) - a(0, H) \leqq b(\delta_0) - a(0, H) = 0 .$$

Thus, we can find a $t \in (t_0, t_0+T]$ so that $V(t^*, x(t^*, t_0, x_0)) < a(0, \varepsilon)$.

For some $\varepsilon > 0$, we assume that a solution $y(t, t_0, x_0)$ for $x_0 \in D$ satisfies $|y(t, t_0, x_0)| = \varepsilon$ at some $t > t^*$. Then there exists $t' > t^*$ such that $|y(t', t_0, x_0)| = \varepsilon$ and $|y(t, t_0, x_0)| < \varepsilon$ for all $t \in (t^*, t')$. From (i), it follows that $a(0, \varepsilon) \leqq a(t', |y(t', t_0, x_0)|) \leqq V(t', x(t', t_0, x_0)) \leqq V(t^*, x(t^*, t_0, x_0)) < a(0, \varepsilon)$. This is a contradiction, and hence, if $t^* \leqq t_0 + T \leqq t$, then $|y(t, t_0, x_0)| < \varepsilon$. This shows that the solution $x = 0$ is asymptotically y-stable uniformly in x_0 from the region D. Thus, the proof is completed.

References

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