Some Generalization of Stability Theorem by Utilizing Vector Valued Liapunov Function

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1. Introduction

It is well-known that A.M.Liapunov has discussed the stability of solutions of the system of differential equations by utilizing a scalar function satisfying some conditions.

Liapunov's second method is a very useful and powerful instrument in discussing the stability of the system of differential equations. Its power and usefulness lie in the fact that the criterion of stability can be decided from the differential equations without any knowledge of their solutions. However, it is very difficult to find the Liapunov function satisfying certain conditions. Therefore, it is important to obtain a weak sufficient condition for a stability theorem.

In [1], V.M. Matrosov obtained the test for stability with the simultaneous use of the several Liapunov functions. In this case, each function satisfies less rigid conditions than the one function occurring in the corresponding theorem of Liapunov's second method.

His works show that the use of the several Liapunov functions can lead to a more flexible theorem.

In this paper, we will state some generalization of Matrosov's stability theorem.

2. Definitions and Notations

Let I denote the interval $0 \leqq t < \infty$ and R^n denote Euclidean n-space.

For $x \in R^n$, let $\| x \|$ be any norm of x, and we shall denote by S_H the set of x such that $\| x \| < H$, $H > 0$.

We consider a system of differential equations

(1) $$\frac{dx}{dt} = f(t,x).$$

Suppose that $f(t,x)$ is continuous in (t,x) on $I \times \bar{S}_H$ and that $f(t,x)$ satisfies the Lipschitz condition with respect to x, moreover $f(t,0) \equiv 0$.

Let us consider the real functions $V_1(t,x)$, $V_2(t,x)$, ……, $V_k(t,x)$ which satisfy that $V_i(t,x)$ is a scalar continuous function in (t,x) on $I \times \bar{S}_H$ $(i=1, \cdots\cdots, k)$ and let $\dot{V}_i(t,x)$ be the derivatives with respect to t, taken relative to the equations (1), and let $V_i(t,0)=0$, $\dot{V}_i(t,0)=0$.

For the set of these functions $V=(V_1, \cdots\cdots, V_k)$, we introduce the norm

$\|V\| = |V_1| + \cdots\cdots + |V_k|$.

Let R_1 be $\sup\{\|V(t,x)\| : (t,x) \in I \times \bar{S}_H\}$ and G be $\{(t,x): t \in I, \|V(t,x)\| < R_2\}$, where $R_2 > R_1$ or $R_2 = \infty$.

Let the following system of equations be given

(2) $$\frac{dy_i}{dt} = w_i(t, y_1, \cdots\cdots, y_k) \quad (i=1, \cdots\cdots, k),$$

where the functions w_i are real and continuous on G, and $w_i(t,0) = 0$.

We introduce the following definitions.

〔Definition 1.〕 The zero solution of the system (1) is said to be stable if for any $\varepsilon > 0$ and any $t_0 \in I$ there exists $\delta(\varepsilon, t_0) > 0$ such that the inequality $\|x_0\| < \delta(\varepsilon, t_0)$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geqq t_0$, where $x(t, t_0, x_0)$ denote the solution of system (1) satisfying the initial condition $x(t_0, t_0, x_0) = x_0$.

〔Definition 2.〕 The zero solution of the system (1) is said to be uniformly stable if δ of 〔Definition 1.〕 is independent of t_0.

〔Definition 3.〕 The zero solution of the system (1) is said to be asymptotically stable if it is stable and if there exists $\delta_0(t_0) > 0$ such that $\|x_0\| < \delta_0(t_0)$, $x(t, t_0, x_0) \longrightarrow 0$ as $t \longrightarrow \infty$.

〔Definition 4.〕 The zero solution of the system (1) is said to be uniformly attractive if for any $\varepsilon > 0$ and $t_0 \in I$ there exist $\delta_0 > 0$ and $T(\varepsilon) > 0$ such that the inequality $\|x_0\| < \delta_0$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geqq t_0 + T(\varepsilon)$.

〔Definition 5.〕 The zero solution of the system (1) is said to be uniformly asymptotically stable if it is uniformly stable and uniformly attractive.

〔Definition 6.〕 The zero solution of the system (2) is partially stable with respect to $y_1, \cdots\cdots, y_m$ $(1 \leqq m \leqq k)$ if for any $\varepsilon > 0$ and any $t_0 \in I$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\|y_0\| < \delta$ implies $|y_1(t, t_0, y_0)| + \cdots\cdots + |y_m(t, t_0, y_0)| < \varepsilon$ for all $t \geqq t_0$.

〔Definition 7.〕 The zero solution of the system (2) is said to be partially uniformly stable with respect to $y_1, \cdots\cdots, y_m$ if δ of 〔Definition 6.〕 is independent of t_0.

〔Definition 8.〕 The zero solution of the system (2) is said to be partially asymptotically stable with respect to $y_1, \cdots\cdots, y_m$ if it is partially stable with respect to $y_1, \cdots\cdots, y_m$ and if there exists $\delta_0(t_0) > 0$ such that $\|y_0\| < \delta_0(t_0)$ implies $|y_1(t, t_0, y_0)| + \cdots\cdots + |y_m(t, t_0, y_0)| \longrightarrow 0$ as $t \longrightarrow \infty$.

〔Definition 9.〕 The zero solution of the system (2) is said to be partially uniformly asymptotically stable with respect to $y_1, \cdots\cdots, y_m$ if it is partially uniformly stable with resspect to $y_1, \cdots\cdots, y_m$ and partially uniformly attractive, i.e., if any $\varepsilon > 0$ and any $t_0 \in I$ there exixt $\delta_0 > 0$ and $T = T(\varepsilon) > 0$ such that $\|y_0\| < \delta_0$ implies $|y_1(t, t_0, y_0)| + \cdots\cdots + |y_m(t, t_0, y_0)| < \varepsilon$ for all $t \geqq t_0 + T(\varepsilon)$.

〔Definition 10.〕 A scalar function $U(t,x)$ is called positively definite if there exists a continuous increasing function $\varphi(r)$, where $\varphi(0) = 0$, such that $\varphi(\|x\|) \leqq U(t,x)$ holds for all $t \geqq t_0$.

〔Definition 11.〕 A scaler function $U(t,x)$ admits an infinitesimal upper bound if there exists a continuous increasing function $\varphi(r)$, where $\varphi(0) = 0$, such that $U(t,x) \leqq \varphi(\|x\|)$

holds in the neighborhood of the origin and for all $t \geqq t_0$.

3. Preliminary Results

【Theorem A】 *Let there exist functions* $V_1(t,x), \cdots\cdots, V_k(t,x)$ *possessing the following properties in* $I \times \overline{S}_H$:

(i) *the functions* $V_1(t,x) \geqq 0, \cdots\cdots, V_m(t,x) \geqq 0$ $(1 \leqq m \leqq k)$ *and the function* $V_1(t,x) + \cdots\cdots + V_m(t,x)$ *is positively definite*,

(ii) *the derivatives relative to the system* (1) *are* $\dot{V}_i(t,x) = w_i(t,V) + W_i(t,x)(i=1,\cdots,k)$, *where* $W_i(t,x) \leqq 0$ *and are continuous*,

(iii) *each of the functions* $w_i(t,V)$ *is non-decreasing with respect to the functions* $V_1, \cdots\cdots, V_{i-1}, V_{i+1}, \cdots\cdots, V_k$ *in* G.

If the zero solution $y_1=0, \cdots\cdots, y_k=0$ *of the system* (2) *is stable (or asymptotically stable) with respect to* $y_1, \cdots\cdots, y_m$ *under the conditions* $y_{01} \geqq 0, \cdots, y_{0m} \geqq 0$, *where* $y_0 = (y_{01}, \cdots, y_{0k})$, *then the zero solution* $x=0$ *of the system* (1) *is stable (or asymptotically stable).*

【Theorem B】 *If one adds to the hypotheses of* Theorem A *that*

(iv) *the functions* $V_1(t,x), \cdots\cdots, V_k(t,x)$ *admit an infinitesimal upper bound.*

If the stability of the zero solusion of the system (2) *is uniform with respect to* t_0 *(or the asymptotic stability is uniform with respect to* $y_{01}, \cdots, y_{0m}, t_0$*), then the stability of the zero solution of the system* (1) *will be uniform with respect to* t_0 *(or the asymptotic stability will be uniform with respect* to x_0 *and* t_0*).*

For the proof of these theorems, see [1].

【Theorem C】 *If* $\lambda : I \to R$ *is a continuous function, the equation* $\dot{u} = \lambda(t)u$ *has a critical point at the origin, which is stable, uniformly stable or equi-asymptotically stable according to*

$$(\forall\ t_0 \in I)\ (\exists\ A > 0)\ (\forall t \geqq t_0)\ \int_{t_0}^{t} \lambda(s)\,ds \leqq A,$$

$$(\exists A > 0)\ (\forall\ t_0 \in I)\ (\forall\ t \geqq t_0)\ \int_{t_0}^{t} \lambda(s)\,ds \leqq A,$$

$$or\ (\forall\ t_0 \in I)\ \int_{t_0}^{t} \lambda(s)\,ds \to -\infty \text{ as } t \to \infty.$$

See [2].

4. Main Results

Before we state main results, we give the following lemma on differential inequalities of Ważewski.

【Lemma】 *Let us consider the system* (2). *Suppose that each function* $w_i(t,y)$ *of the system* (2) *is non-decreasing with respect to* $y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_k$ *in the some open region* Ω. *Then for any point* $(t_0, y_0) \in \Omega$, *there exist one upper integral* $y^+(t,t_0,y_0)$ *and one lower integral* $y^-(t,t_0,y_0)$ *defined on* $[t_0, \alpha)$.

Moreover, let functions $\varphi_1(t), \cdots, \varphi_k(t)$ *be given continuously differentiable in the interval*

$[t_0, \alpha)$ *such that* $\varphi_i(t_0) = y_{0i}$ $(i=1,\cdots,k)$ *and* $(t, \varphi_1(t), \cdots, \varphi_k(t)) \in \Omega$ *when* $t \in [t_0, \alpha)$.

If $\dfrac{d\varphi_i(t)}{dt} \leqq w_i(t, \varphi_1(t), \cdots, \varphi_k(t))$

$\left(or \quad \dfrac{d\varphi_i(t)}{dt} \geqq w_i(t, \varphi_1(t), \cdots, \varphi_k(t))\right.$,

then $\varphi_i(t) \leqq y_i^+(t, t_0, y_0)$ $(or \ \varphi_i(t) \geqq y_i^-(t, t_0, y_0))$ *for all* $t \in [t_0, \alpha)$ $(i=1,\cdots,k)$.

For the proof, see [6].

【Theorem 1】 *Suppose that there exist the functions* $V_1(t,x), \cdots, V_k(t,x)$ *satisfying the following conditions in* $I \times \bar{S}_H$:

(i) *the functions* $V_1(t,x) \geqq 0, \cdots, V_m(t,x) \geqq 0$ $(1 \leqq m \leqq k)$, *and* $a(t, \|x\|) \leqq V_1(t,x) + \cdots + V_m(t,x)$, *where* $a(t,r)$ *is continuous in* (t,r) *on* $I \times R$ *and* $a(t,r) > 0$ *for* $r \neq 0$, $a(t,0) \equiv 0$,

(ii) *the derivatives relative to the system* (1) *are*

$\dot{V}_i(t,x) = w_i(t,V) + W_i(t,x)$ $(i=1,\cdots,k)$, *where* $W_i(t,x) \leqq 0$ *and are continuous*,

(iii) *each of the functions* $w_i(t,V)$ *is non-decreasing with respect to the functions* $V_1, \cdots, V_{i-1}, V_{i+1}, \cdots, V_k$ *in* G.

If the zero solution $y_1=0, \cdots, y_k=0$ *of the system* (2) *is stable with respect to* $y_1, \cdots, y_m$ *under the conditions* $y_{01} \geqq 0, \cdots, y_{0m} \geqq 0$, *where* $y_0 = (y_{01}, \cdots, y_{0k})$, *then the zero solution* $x=0$ *of the system* (1) *is stable*.

Proof. For any $\varepsilon > 0$ $(0 < \varepsilon < H)$, if we assume that
$\inf\{V_1(t,x) + \cdots + V_m(t,x) : t \in I, \|x\| \geqq \varepsilon\} = 0$, then there exists
$(t_1, x_1) \in I \times \bar{S}_H$ such that $\|x_1\| \geqq \varepsilon$ and $V_1(t_1, x_1) + \cdots + V_m(t_1, x_1) = 0$.
On the other hand, by (i) for $t_1 \geqq 0$, $\|x_1\| \geqq \varepsilon$, we obtain
$V_1(t_1, x_1) + \cdots + V_m(t_1, x_1) \geqq a(t_1, \|x_1\|) > 0$. This is a contradiction. Therefore we have
$0 < \inf\{V_1(t,x) + \cdots + V_m(t,x) : t \in I, \|x\| \geqq \varepsilon\} \leqq R_1$.
Therefore, if we take a positive number $\rho(\varepsilon)$ such that
$\rho(\varepsilon) < \inf\{V_1(t,x) + \cdots + V_m(t,x) : t \in I, \|x\| \geqq \varepsilon\}$, then we have $\|x\| < \varepsilon$ when $t \geqq 0$ and $V_1(t,x) + \cdots + V_m(t,x) \leqq \rho(\varepsilon)$.

According to Wazewski's lemma, the existence of the upper integral $y^+(t, t_0, y_0)$ of the system (2) is guaranted. By virtue of the assumption of stability for the zero solution of system (2) with respect to $y_1, \cdots, y_m$ when $y_{01} \geqq 0, \cdots, y_{0m} \geqq 0$ along $\rho(\varepsilon)$ for $t_0 \geqq 0$, we can find a positive number $\delta_0 = \delta_0(\rho(\varepsilon), t_0)$ $(0 < \delta_0 < \rho(\varepsilon) < R_1)$ such that $|y_1^+(t, t_0, y_0)| + \cdots + |y_m^+(t, t_0, y_0)| < \rho(\varepsilon)$ for all $t \geqq t_0$ when $|y_{01}| + \cdots + |y_{0k}| \leqq \delta_0$, $y_{01} \geqq 0, \cdots, y_{0m} \geqq 0$. Since $V = (V_1, \cdots, V_k)$ is continuous on $I \times \bar{S}_H$, for $\delta_0 > 0$ and $t_0 \geqq 0$ there is $\eta = \eta(\delta_0, t_0) > 0$ such that

(3) $\quad |V_1(t_0, x_0)| + \cdots + |V_k(t_0, x_0)| \leqq \delta_0$ for all $\|x_0\| < \eta$.

Let us show that for any $x_0 \in S_\eta$ and any $t_0 \in I$, $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geqq t_0$. If we assume that this is not true, there exist $x^* \in S_\eta$ and $t^* > t_0$ such that $x(t^*, t_0, x_0^*) = \varepsilon$, but $\|x(t, t_0, x_0^*)\| < \varepsilon$ when $t \in [t_0, t^*)$.

Let us set $y_{0i}^* = V_i(t_0, x_0^*)$ $(i=1,\cdots,k)$. Then by (3),

$|y^*_{01}| + \cdots + |y^*_{0k}| = |V_1(t_0, x^*_0)| + \cdots + |V_k(t_0, x^*_0)| \leq \delta_0$,
but according to the condition (i)
$y^*_{01} = V_1(t_0, x^*_0) \geqq 0, \cdots, y^*_{0m} = V_m(t_0, x^*_0) \geqq 0$, and by choosing δ_0
$|y_1^+(t, t_0, y^*_0)| + \cdots + |y^+_m(t, t_0, y^*_0)| < \rho(\varepsilon)$ for all $t \in [t_0, t^*]$.

Let us consider the functions $V_i(t, x(t, t_0, x^*_0))$ $(i=1,\cdots,k)$ which are continuously differentiable with respect to t in the interval $[t_0, t^* + \triangle t)$, where $\triangle t > 0$ is sufficiently small. By virtue of the condition (ii),

$$\frac{dV_i(t, x(t, t_0, x^*_0))}{dt} \leqq w_i(t, V(t, x(t, t_0, x^*_0))) \quad \text{for all } t \in [t_0, t^* + \triangle t).$$

Therefore, by applying Wazewski's lemma, we get

$V_i(t, x(t, t_0, x^*_0)) \leqq y_i^+(t, t_0, y^*_0)$ $(i=1,\cdots,k)$ for all $t \in [t_0, t^*]$, and consequently

$$\sum_{i=1}^{m} V_i(t, x(t, t_0, x^*_0)) \leqq \sum_{i=1}^{m} |y^+_0(t, t_0, y^*_0)| < \rho(\varepsilon).$$

Therefore we have

$\|x(t, t_0, x^*_0)\| < \varepsilon$ for all $t \in [t_0, t^*]$ and in particular $\|x(t^*, t_0, x^*_0)\| < \varepsilon$,
which contradicts the assumption we have made. This proves the stability of the zero solution of the system (1).

【Theorem 2】 *Under the same assumptions of Theorem* 1, *moreover we suppose that*
(iv) $|V_i(t, x)| \leqq b_i(\|x\|)$, *where the functions* $b_i(r)$ *are continuous, increasing and positively definite,* $(i=1,\cdots,k)$.
If the zero solution of the system (2) *is uniformly stable with respect to* $y_1, \cdots, y_m$ *under the conditions* $y_{01} \geqq 0, \cdots, y_{0m} \geqq 0$, *then the zero solution of the system* (1) *is uniformly stable.*

Proof. By the same argument as in Theorem 1, for any $\varepsilon > 0$, we can choose $\rho(\varepsilon) > 0$ such that $\rho(\varepsilon) < \inf\{V_1(t, x) + \cdots + V_m(t, x) : t \in I, \|x\| \geqq \varepsilon\}$ and then we have $\|x\| < \varepsilon$ when $t \geqq 0$ and $V_1(t, x) + \cdots + V_m(t, x) \leqq \rho(\varepsilon)$.

Since the zero solution of (2) is uniformly stable with respect to $y_1, \cdots, y_m$ under the conditions $y_{01} \geqq 0, \cdots, y_{0m} \geqq 0$ for any $t_0 \in I$, there exixts $\delta = \delta(\rho(\varepsilon)) > 0$ such that $|y_1^+(t, t_0, y_0)| + \cdots + |y^+_m(t, t_0, y_0)| < \rho(\varepsilon)$ for all $t \geqq t_0$, if $|y_{01}| + \cdots + |y_{0k}| \leqq \delta$ and $y_{01} \geqq 0, \cdots, y_{0m} \geqq 0$. By the condition (iv), there exists $\eta = \eta(\delta(\varepsilon)) > 0$ such that $b(\eta) < \delta$, therefore, if $\|x_0\| < \eta$, $|V_1(t_0, x_0)| + \cdots + |V_k(t_0, x_0)| < \delta$ for any $t_0 \in I$.

By the same argument as the one in Theorem 1, it can be proved that for any $t_0 \in I$ and any $x_0 \in S_\eta$, $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geqq t_0$, which shows the uniform stability of the zero solution of the system (1).

Next, we shall discuss the theorems on asymptotic stability and uniformly asymptotic stability of the zero solution of the system (1).

【Theorem 3】 *If the condition* (i) *in Theorem* 1 *is replaced by*
(i)′ $V_1(t, x) \geqq 0, \cdots, V_m(t, x) \geqq 0$ $(1 \leqq m \leqq k)$, *and*
$a(t, \|x\|) \leqq V_1(t, x) + \cdots + V_m(t, x)$, *where* $a(t, r)$ *is continuous in* (t, r) *on* $I \times R$, $a(t, r) > 0$ *for* $r \neq 0$, $a(t, 0) \equiv 0$ *and* $a(t, p(t)) \to 0$ *implies* $p(t) \to 0$ *as* $t \to \infty$.

If the zero solution $y_1 = 0, \cdots, y_k = 0$ *of the system* (2) *is asymptotically stable with respect to* $y_1, \cdots, y_m$ *under the conditions* $y_{01} \geq 0, \cdots, y_{0m} \geq 0$, *then the zero solution of the system* (1) *is asymptotically stable.*

Proof. By Theorem 1, the zero solution of (1) is stable. Since the zero solution of (2) is asymptotically stable with respect to $y_1, \cdots, y_m$ under the conditions $y_{01} \geq 0, \cdots, y_{0m} \geq 0$, for any $\varepsilon > 0$ and any $t_0 \in I$ there exists $T = T(\varepsilon, t_0, y_0) > 0$ such that

$\sum_{i=1}^{m} | y_i^{+}(t, t_0, y_0) | < \varepsilon$ for all $t \geq t_0 + T$ when $y_0 \in \bar{S}_\delta$ and $x_0 \in \bar{S}_\eta$, where δ_0 and η are these in Theorem 1. According to (ii), by applying Wazewski's lemma, we get

$$\sum_{i=1}^{m} V_i(t, x(t, t_0, x_0)) < \varepsilon \quad \text{for all} \quad t \geq t_0 + T.$$

Thus, for $\| x_0 \| \leq \eta$ we have $\lim_{t \to \infty} \sum_{i=1}^{m} V_i(t, t_0, x_0)) = 0$.

According to (i), since $a(t, \| x(t, t_0, x_0) \|) \leq V(t, x(t, t_0, x_0))$, we have $\lim_{t \to \infty} \| x(t, t_0, x_0) \| = 0$. Therefore the zero solution of (1) is asymptotically stable.

【Theorem 4】 *Suppose that there exist the functions* $V_1(t, x), \cdots, V_k(t, x)$ *defined on* $I \times \bar{S}_H$ *satisfying the following conditions* :

(i) $V_1(t, x) \geq 0, \cdots, V_m(t, x) \geq 0$ $(1 \leq m \leq k)$, *and* $a(t, \| x \|) \leq V_1(t, x) + \cdots + V_m(t, x)$, *where* $a(t, r)$ *is continuous in* (t, r) *on* $I \times R$ *and* $a(t, r) > 0$ *for* $r \neq 0$, $a(t, 0) \equiv 0$,

(ii) *the derivatives relative to the system* (1) *are*
$\dot{V}_i(t, x) = w_i(t, V) + W_i(t, x)$ $(i = 1, \cdots, k)$, *where* $W_i(t, x) \leq 0$ *and are continuous,*

(iii) *each of the functions* $w_i(t, V)$ *is non-decreasing with respect to the functions* $V_i, \cdots, V_{i-1}, V_{i+1}, \cdots, V_k$ *in* G,

(iv) $| V_i(t, x) | \leq b_i(\| x \|)$ $(i = 1, \cdots, k)$, *where* $b_i(r)$ is *continuous, increasing and positively definite.*

If the zero solution $y_1 = 0, \cdots, y_k = 0$ *of the system* (2) *is uniformly asymptotically stable with respect to* $y_1, \cdots, y_m$ *under the conditions* $y_{01} \geq 0, \cdots, y_{0m} \geq 0$, *then the zero solution of the system* (1) *is uniformly asymptotically stable.*

Proof. By Theorem 2, the zero solution of (1) is uniformly stable. We shall prove that the zero solution of (1) is uniformly attractive. By the same argument as in Theorem 1, it can be proved that for any $\varepsilon > 0$ if we take $\rho(\varepsilon) > 0$ such that $\rho(\varepsilon) < \inf \{ V_1(t, x) + \cdots + V_m(t, x) : t \in I, \| x \| \geq \varepsilon \}$ then we have $\| x \| < \varepsilon$ when $t \geq 0$ and $V_1(t, x) + \cdots + V_k(t, x) \leq \rho(\varepsilon)$.

Since the zero solution of (2) is uniformly attractive, for any $\varepsilon > 0$ and any $t_0 \in I$, there exist $\delta > 0$ and $T = T(\varepsilon) > 0$ such that if

$| y_{01} | + \cdots + | y_{0k} | \leq \delta$ and $y_{01} \geq 0, \cdots, y_{0m} \geq 0$, then $\sum_{i=1}^{m} | y_i^{+}(t, t_0, y_0) | < \rho(\varepsilon)$ for all $t \geq t_0 + T$. According to the condition (iv), we can choose $\delta_1 > 0$ such that $b(\delta_1) < \delta$, where $b(r) = b_1(r) + \cdots + b_k(r)$.

For any $t_0 \in I$ and $\|x_0\| \leqq \delta_1$, we have $\sum_{i=1}^{k} |V_i(t_0, x_0)| < \delta$.

Let us assume that for t_0 and $\|x_0\| \leqq \delta_1$, there exists $t^* > t_0 + T$ such that $\|x(t^*, t_0, x_0)\| = \varepsilon$, $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \in [t_0 + T, t^*)$.

We put $y_{0i} = V_i(t_0, x_0)$ $(i = 1, \cdots, k)$. By the choosing δ,

$\sum_{i=1}^{m} | y_i^\dagger(t, t_0, y_0) | < \rho(\varepsilon)$ for all $t \in [t_0 + T, t^*]$ because

$|y_{01}| + \cdots + |y_{0k}| = |V_1(t_0, x_0)| + \cdots + |V_k(t_0, x_0)| < \delta$, $y_{01} = V_1(t_0, x_0) \geqq 0$
$\cdots, y_{0m} = V_m(t_0, x_0) \geqq 0$.

By virtue of (ii), $\sum_{i=1}^{m} V_i(t, x(t, t_0, x_0)) \leqq \sum_{i=1}^{m} | y_i^\dagger(t, t_0, y_0)|$
by applying Wazewski's lemma, and consequently $\sum_{i=1}^{m} V_i(t, x(t, t_0, x_0)) < \rho(\varepsilon)$
for all $t \in [t_0 + T, t^*]$. Therefore we have $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \in [t_0 + T, t^*]$, and in particular $\|x(t^*, t_0, x_0)\| < \varepsilon$, which contradicts the assumption we have made. Therefore the zero solution of (1) is uniformly attractive. This completes the proof.

5. Example

Let us consider the equation

$$\frac{dx}{dt} = (\sin t + e^{-t})x + (\sin t - e^{-t})y - (x^3 + xy^2)\sin^2 t$$

(E)

$$\frac{dy}{dt} = (\sin t - e^{-t})x + (\sin t + e^{-t})y - (x^2 y + y^3)\sin^2 t$$

and define the functions $V_1(t, x, y)$ and $V_2(t, x, y)$ as follows :

$$V_1(t, x, y) = \frac{1}{2} e^{-t}(x + y)^2, \quad V_2(t, x, y) = \frac{1}{2} e^{-t}(x - y)^2.$$

This function $V_1(t, x, y)$ does not satisfy the conditions of our theorem, because $V_1(t, x, y) = 0$ as $x = -y$. The function $V_2(t, x, y)$ doesn't, either.

However if we consider the vector $V = (V_1, V_2)$, then we see that this function V satisfies our results as follows.

1°) It is clear that $V_1(t, x, y) \geqq 0$, $V_2(t, x, y) \geqq 0$ and $V_1 + V_2 = e^{-t}(x^2 + y^2)$. We set $a(t, r) = e^{-t} r^2$. This function satisfies that $a(t, r) > 0$ for $r \neq 0$, $a(t, 0) \equiv 0$. The inequalities $V_1 = \frac{1}{2} e^{-t}(x + y)^2 \leqq x^2 + y^2$ and $V_2 = \frac{1}{2} e^{-t}(x - y)^2 \leqq x^2 + y^2$ imply that V_1 and V_2 have the infinitesimal upper bound.

2°) By the simple calculation, we have $\dot{V}_1 = w_1(t, V_1, V_2) + W_1(t, x, y)$, where $w_1(t, V_1, V_2) = 4 \sin t \cdot V_1$ which is non-decreasing with respect to V_2 and $W_1(t, x, y) = -(x + y)^2 (x^2 + y^2) e^{-t} \sin^2 t - \frac{1}{2} e^{-t}(x + y)^2$ which is negative and continuous in t, x and y.

Also we have $\dot{V}_2 = w_2(t, V_1, V_2) + W_2(t, x, y)$, where $w_2(t, V_1, V_2) = 4 e^{-t} V_2$ which is non-decreasing with respect to V_1 and

$W_2(t,x,y) = -(x-y)^2(x^2+y^2)e^{-t}\sin^2 t - \frac{1}{2}e^{-t}(x-y)^2$ which is negative and continuous in t, x and y.

3°) Consider the system

$$(E)' \quad \frac{dy_1}{dt} = 4\sin t \cdot y_1, \qquad \frac{dy_2}{dt} = 4e^{-t} y_2 .$$

For any $t_0 \geqq 0$, we have $\int_{t_0}^{t} 4\sin s\, ds \leqq 4\sqrt{2}$ and $\int_{t_0}^{t} 4e^{-s} ds \leqq 4$.
Therefore, by Theorem C, the zero solution of (E)′ is uniformly stable.
Applying our theorem, we have found that the zero solution of (E) is uniformly stable.

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