Partial Stability Theorems by the Comparison Principle

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1. Introduction

Explicit solutions of differential equations are often wholly out of the question.

Our purpose in solving most of the differential equations is to see their qualities rather than the concrete expressions of their solutions such as series or elementary functions.

Liapunov's second method enables us to decide stability and boundedness from the differential equations without any knowledge of their solutions. (cf. [1], [5], [6])

C.Corduneau [2] and H.A.Antosiewitz [3] observed that the Liapunov's second method depends basically on the fact that a function $u(t)$ satisfying the inequality $\dot{u}(t) \leqslant w(t, u(t))$ $(u(t_0) \leqslant r_0)$ is majorized by the maximal solution of the scalar differential equation $\dot{r} = w(t, r), r(t_0) = r_0$.

In this paper, using this method, that is, by the comparison with a scalar differential equation, we extend the Liaqunov's second method to various partial stability theorems of the system of the differential equation.

2. Definitions and Notations

Let I denote the interval $0 \leqslant t < \infty$ and R^n denote Euclidean n-space. For $x \in R^n$, let $\| x \|$ be any norm of x. Let $n > 0$ and $m > 0$ be two integers, and consider the two continuous functions $f : I \times D \times R^m \to R^n$, $g : I \times D \times R^m \to R^m$, where D is a domain such that $\| x \| < h \leqslant \infty, \| y \| < \infty (h > 0)$. We assume that $f(t, 0, 0) \equiv 0$ and $g(t, 0, 0) \equiv 0$ for any $t \in I$ and further that f and g are smooth enough in order that, through every point of $I \times D \times R^m$, there passes one and only one solution of the differential system

$$\frac{dx}{dt} = f(t, x, y), \quad \frac{dy}{dt} = g(t, x, y). \tag{1}$$

Let the solutions of (1) be y-prolongeable. A solution through a point (t_0, x_0, y_0) in $I \times D \times R^m$ will be denoted by a form as $(x(t, t_0, x_0, y_0), \ y(t, t_0, x_0, y_0))$.

Suppose that there exists a scalar differential equation

$$\frac{dv}{dt} = w(t, v), \tag{2}$$

where w is the continuous function defined on $I \times R$, $w(t, 0) \equiv 0$ and satisfies the conditions which ensure the existence and uniqueness of the solutions of (2). A solution through a point (t_0, v_0) in $I \times I$ will be denoted by such a form as $v(t, t_0, v_0)$.

We introduce the following definitions.

【Definition 1】 The solution $v = 0$ of (2) is stable, if for any $\varepsilon > 0$ and any $t_0 \in I$, there exists a $\delta(t_0, \varepsilon) > 0$ such that if $v_0 < \delta(t_0, \varepsilon)$, we have $v(t, t_0, v_0) < \varepsilon$ for all $t \geqslant t_0$.

【Definition 2】 The solution $v = 0$ of (2) is uniformly stable, if the δ above is independent of t_0.

【Definition 3】 The solution $v = 0$ of (2) is asymptotically stable, if it is stable and if there exists a $\delta_0(t_0) > 0$ such that if $v_0 < \delta_0(t_0)$, $v(t, t_0, v_0) \to 0$ as $t \to \infty$.

【Definition 4】 The solution $v = 0$ of (2) is equi-asymptotically stable, if it is stable and if given any $\varepsilon > 0$ and any $t_0 \in I$, there exist a $\delta_0(t_0) > 0$ and a $T(t_0, \varepsilon) > 0$ such that if

$v_0 < \delta_0(t_0)$, $v(t, t_0, v_0) < \varepsilon$ for all $t \geqslant t_0 + T(t_0, \varepsilon)$.

【Definition 5】 The solution $v = 0$ of (2) is uniformly asymptotically stable, if it is uniformly stable and if given any $\varepsilon > 0$ and any $t_0 \in I$, there exist a $\delta_0 > 0$ and a $T(\varepsilon) > 0$ such that if $v_0 < \delta_0$, $v(t, t_0, v_0) < \varepsilon$ for all $t \geqslant t_0 + T(\varepsilon)$.

【Definition 6】 The solution $x = 0$, $y = 0$ of (1) is partially stable with respect to x if for any $\varepsilon > 0$ and any $t_0 \in I$, there exists a $\delta(t_0, \varepsilon) > 0$ such $\| x_0 \| + \| y_0 \| < \delta$ implies $\| x(t, t_0, x_0, y_0) \| < \varepsilon$ for all $t \geqq t_0$.

【Definition 7】 The solution $x = 0$, $y = 0$ of (1) is partially uniformly stable with respect to x, if the δ above is independent of t_0.

【Definition 8】 The solution $x = 0$, $y = 0$ of (1) is partially asymptotically stable with respect to x, if it is partially stable with respect to x and if there exists a $\delta_0(t_0) > 0$ such that $\| x_0 \| + \| y_0 \| < \delta_0$ implies $x(t, t_0, x_0, y_0) \to 0$ as $t \to \infty$.

【Definition 9】 The solution $x = 0$, $y = 0$ of (1) is partially uniformly asymptotically stable with respect to x, if it is partially uniformly stable with respect to x and if there exist a δ_0 and a $T(\varepsilon) > 0$ such that $\| x_0 \| + \| y_0 \| < \delta_0$ implies $\| x(t, t_0, x_0, y_0) \| < \varepsilon$ for all $t \geqslant t_0 + T(\varepsilon)$.

【Definition 10】 For a continuous function $V : I \times D \times R^m \to R$, we define the function

$$V'_{(1)}(t, x, y) = \limsup_{h \to +0} \frac{1}{h} \{ V(t+h, x+hf(t, x, y), y+hg(t, x, y)) - V(t, x, y) \}.$$

In case $V(t, x, y)$ has continuous partial derivatives of the first order, it is evident that

$$V'_{(1)}(t, x, y) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t, x, y) + \frac{\partial V}{\partial y} \cdot g(t, x, y),$$

where " . " denote a scalar product.

3. Preliminary Results

【Theorem 1】 Let $V : I \to R$ be such that,

(i) $V(t_0) \leqslant v_0$,

(ii) $\dot{V}(t) \leqslant w(t, V(t))$ on I,

then $V(t) \leqslant v(t, t_0, v_0)$ on I.

For the proof, see [1].

【Theorem 2】 (C.Corduneanu [1964]) Suppose there exists a continuously differentiable function $V : I \times D \times R^m \to R$, such that

(i) $V(t, 0, 0) \equiv 0$,

(ii) $a(\| x \|) \leqslant V(t, x, y)$, where the function $a(r)$ is continuous on I, $a(0) = 0$, $a(r) > 0$ for $r \neq 0$ and increases monotonically,

(iii) $\dot{V}(t, x, y) \leqslant w(t, V(t, x, y))$,

then

(a) stability of the solution $v = 0$ of (2) implies stability with respect to x of the solution $x = 0$, $y = 0$ of (1);

(b) asymptotic stability of the solution $v = 0$ of (2) implies equi-asymptotic stability with respect to x of the solution $x = 0$, $y = 0$ of (1), provided the solutions of (1) do not approach ∞ in a finite time;

if moreover,

(iv) $V(t, x, y) \leqslant b(\| x \| + \| y \|)$, where the function $b(r)$ is continuous in I, $b(0) = 0$, $b(r) > 0$ for $r \neq 0$ and increases monotonically, then

(c) uniform stability of the solution $v=0$ of (2) implies uniform stability with respect to x of the solution $x=0$, $y=0$ of (1);

(d) uniformly asymptotic stability of the solution $v=0$ of (2), along with the same assumptions as in(b) for the existence of solutions, implies uniformly asymptotic stability with respect to x of the solution $x=0$, $y=0$ of (1). For the proof, see [3].

4 . Main Results

【Theorem 3】 If there exists a continuous function $V: I\times D\times R^m \to R$ such that

(i) $V(t,0,0)\equiv 0$,

(ii) $a(t,\|x\|)\leqq V(t,x,y)$, where the function $a(t,r)$ is continuous in (t,r) on $I\times R$, $a(t,0)\equiv 0$, $a(t,r)>0$ for $r\neq 0$ and increases monotonically with respect to t for each fixed r,

(iii) $V'_{(1)}(t,x,y)\leqq w(t,V(t,x,y))$,

and if the solution $v=0$ of (2) is stable, then the solution $x=0$, $y=0$ of (1) is partially stable with respect to x.

[Proof] From the stability of $v=0$, it follows that for any $\varepsilon>0$ and any $t_0\in I$, there exists a $\eta(t_0,\varepsilon)>0$ such that if $v_0<\eta(t_0,\varepsilon)$, then $v(t,t_0,v_0)<a(t_0,\varepsilon)$ for all $t\geqq t_0$. Since V is continuous, we shall see the existence of a $\delta(t_0,\varepsilon)$ such that if $\|x_0\|+\|y_0\|<\delta(t_0,\varepsilon)$, then $V(t_0,x_0,y_0)<v_0$. By the comparison principle, we have

$$V(t,x(t,t_0,x_0,y_0),y(t,t_0,x_0,y_0))\leqq v(t,t_0,v_0).$$

Now, suppose that a solution $(x(t,t_0,x_0,y_0),\ y(t,t_0,x_0,y_0))$ of (1) such that $\|x_0\|+\|y_0\|<\delta(t_0,\varepsilon)$ satisfies $\|x(t_1,t_0,x_0,y_0)\|=\varepsilon$ at some $t_1>t_0$. Then we have

$$a(t_1,\varepsilon)=a(t_1,\|x(t_1,t_0,x_0,y_0)\|)\leqq V(t_1,x(t_1,t_0,x_0,y_0),\ y(t_1,t_0,x_0,y_0))$$
$$\leqq v(t_1,t_0,v_0)<a(t_0,\varepsilon).$$

This is a contradiction, because $a(t,r)$ increases monotonically with respect to t for each fixed r. Hence, if $\|x_0\|+\|y_0\|<\delta(t_0,\varepsilon)$, then $\|x(t,t_0,x_0,y_0)\|<\varepsilon$ for all $t\geqq t_0$, that is, the solution $x=0$, $y=0$ of (1) is partially stable with respect to x.

【Theorem 4】 If there exists a continuous function $V: I\times D\times R^m \to R$ such that

(i) $V(t,0,0)\equiv 0$,

(ii) $a(t,\|x\|)\leqq V(t,x,y)\leqq b(\|x\|+\|y\|)$, where the function $a(t,r)$ is continuous in (t,r) on $I\times R$, $a(t,0)\equiv 0$, $a(t,r)>0$ for $r\neq 0$ and increases monotonically with respect to t for each fixed r, and the function $b(r)$ is continuous on I, $b(0)=0$, and increases monotonically,

(iii) $V'_{(1)}(t,x,y)\leqq w(t,V(t,x,y))$,

and if the solution $v=0$ of (2) is uniformly stable, then the solution $x=0$, $y=0$ of (1) is partially uniformly stable with respect to x.

[Proof] From the uniform stability of $v=0$, it follows that, for any $\varepsilon>0$ and any $t_0\in I$, there exists a $\delta(\varepsilon)>0$ such that if $v_0<\delta(\varepsilon)$, then $v(t,t_0,v_0)<a(0,\varepsilon)$ for all $t\geqq t_0$.

Let $\eta(\varepsilon)=b^{-1}(\delta(\varepsilon))$. Suppose that a solution $(x(t,t_0,x_0,y_0),\ y(t,t_0,x_0,y_0))$ of (1) such that $\|x_0\|+\|y_0\|<\eta(\varepsilon)$ satisfies $\|x(t_1,t_0,x_0,y_0)\|=\varepsilon$ at some $t_1>t_0$. Then we have

$$V(t_0,x_0,y_0)\leqq b(\|x_0\|+\|y_0\|)<b(\eta(\varepsilon))=b(b^{-1}(\delta(\varepsilon)))=\delta(\varepsilon).$$

If we set $v_0=V(t_0,x_0,y_0)$, then we have

$$a(t_1,\varepsilon)=a(t_1,\|x(t_1,t_0,x_0,y_0)\|)\leqq V(t_1,x(t_1,t_0,x_0,y_0),\ y(t_1,t_0,x_0,y_0))$$
$$\leqq v(t_1,t_0,v_0)<a(0,\varepsilon).$$

This is a contradiction, because $a(t,r)$ increases monotonically with respect to t for each fixed r. Hence, if $\|x_0\|+\|y_0\|<\eta(\varepsilon)$, then $\|x(t,t_0,x_0,y_0)\|<\varepsilon$ for all $t\geqq t_0$, that is, the solution $x=0$,

$y=0$ of (1) is partially uniformly stable with respect to x.

【Theorem 5】 If there exists a continuous function $V: I\times D\times R^m\rightarrow R$ such that

(i) $V(t, 0, 0)\equiv 0$,

(ii) $a(t, \|x\|)\leqslant V(t,x,y)$, where the function $a(t,r)$ is continuous in (t,r) on $I\times R$, $a(t, 0)\equiv 0$, $a(t,r)>0$ for $r\neq 0$ and increases monotonically with respect to t for each fixed r and $a(t, s(t))\rightarrow 0$ implies $s(t)\rightarrow 0$ as $t\rightarrow\infty$,

(iii) $V'_{(1)}(t,x,y)\leqslant w(t, V(t,x,y))$,

and if the solution $v=0$ of (2) is asymptotically stable, then the solution $x=0$, $y=0$ of (1) is partially asymptotically stable with respect to x.

〔Proof〕 By Theorem 3, the solution $x=0$, $y=0$ of (1) is partially stable with respect to x. From the asymptotic stability of $v=0$, for any $t_0\in I$, there exists a $\delta'_0(t_0)>0$ such that if $v_0<\delta'_0(t_0)$, then $\lim_{t\rightarrow\infty} v(t,t_0,v_0)=0$. By the continuity of V, there exists a $\delta_0(t_0)>0$ such that if $\|x_0\|+\|y_0\|<\delta_0(t_0)$, then $V(t_0,x_0,y_0)\leqslant v_0$. Therefore we have

$$a(t, \|x(t,t_0,x_0,y_0)\|)\leqslant V(t, x(t,t_0,x_0,y_0), y(t,t_0,x_0,y_0))\leqslant v(t,t_0,v_0),$$

and $\|x(t,t_0,x_0,y_0)\|\rightarrow 0$ as $t\rightarrow\infty$.

Thus, we see that the solution $x=0$, $y=0$ of (1) is partially asymptotically stable with respect to x.

【Theorem 6】 Under the assmption in Theorem 4, if the solution $v=0$ of (2) is uniformly asymptotically stable, then the solution $x=0$, $y=0$ of (1) is partially uniformly asymptotically stable with respect to x.

〔Proof〕 By Theorem 4, the solution $x=0$, $y=0$ is partially uniformly stable with respect to x. From the uniformly asymptotic stability of $v=0$, it follows that for any $\varepsilon>0$ and any $t_0\in I$, there exist an η_0 and a $T(\varepsilon)>0$ such that if $v_0<\eta_0$, then $v(t,t_0,v_0)<a(0,\varepsilon)$ for all $t\geqslant t_0+T(\varepsilon)$. By assumption, there exists a $\delta_0>0$ such that $b(\delta_0)<\eta_0$. Suppose that a solution $(x(t,t_0,x_0,y_0), y(t,t_0,x_0,y_0))$ of (1) such that $\|x_0\|+\|y_0\|<\delta_0$ satisfies $\|x(t_1,t_0,x_0,y_0)\|=\varepsilon$ at some $t_1>t_0+T(\varepsilon)$. Then we have $V(t_0,x_0,y_0)\leqslant b(\|x_0\|+\|y_0\|)\leqslant b(\delta_0)<\eta_0$.

If we set $v_0=V(t_0,x_0,y_0)$, then we have

$$a(t_1,\varepsilon)=a(t_1, \|x(t_1,t_0,x_0,y_0)\|)\leqslant V(t_1, x(t_1, x(t_1,t_0,x_0,y_0), y(t_1,t_0,x_0,y_0))$$
$$\leqslant v(t_1,t_0,v_0)<a(0,\varepsilon).$$

This is a contradiction, because $a(t,r)$ increases monotonically with respect to t for each fixed r. Hence, if $\|x_0\|+\|y_0\|<\delta_0$, then $\|x(t,t_0,x_0,y_0)\|<\varepsilon$ for all $t\geqslant t_0+T(\varepsilon)$, that is, the solution $x=0$, $y=0$ of (1) is partially uniformly asymptotically stable with respect to x.

References

〔 1 〕 N.Rouche, P.Habets and M.Laloy : Stability theory by Liapunov's direct method, Springer-Verlag, 1977.

〔 2 〕 C.Corduneanu : The applications of differential inequlities to stability theory, An.Sti.Univ. Al. I.Cuza, Iasi Sect. I a Mat., 6, 1960, 47-58.

〔 3 〕 C.Cordeneanu : Sur la stabilité partielle, Revue Roumaine de Math. Pures et Appl. 9, 1964, 229-236.

〔 4 〕 H.A.Antosiewicz : An inequality for approximate solutions of ordinary differential equations, Math. Z. 8, 1962, 44-52.

〔 5 〕 T.Yoshizawa : Stability theory and the existence of periodic solutions and almost periodic solutions, Springer-Verlag, 1975.

〔 6 〕 T.Yoshizawa : Stability theory by Liapunov's second method, Math.Soc. Japan, 1966.

〔 7 〕 S.Seino, M.Kudo and M.Aso : On partial stability of solutions of a system of ordinary differential equations, Reserch Reports of Akita Technical College, No. 16, 1981, 122−125.

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