Sasakian Structures for the Generalized Brieskorn Manifolds

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1. Intoroduction.

Recently, T. Takahashi [4] proved that every Brieskorn manifold admits many Sasakian structures. In this paper, we prove that every generalized Brieskorn manifold admits many Sasakian structures.

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2. Sasakian manifolds.

(M, ϕ, ξ, η, g) is called a Sasakian manifold when the following relations hold for the structure tensor fields, η is a 1 —form, ξ is a vector field, ϕ is a (1, 1)—tensor field and g is a Riemanian metric on M such that

$$(1)\quad \begin{aligned} &\eta(\xi) = 1 \\ &\phi^2(X) = -X + \eta(X)\xi \\ &d\eta(X, Y) = g(\phi(X), Y) \qquad g(\xi, \xi) = 1 \\ &N(X, Y) = [X, Y] + \phi[\phi X, Y] + \phi[X, \phi Y] - [\phi X, \phi Y] - (X\eta(Y) - Y\eta(X))\xi = 0 \end{aligned}$$

for any vector fields X and Y on M.

Theorem A. [4] Let (M, ϕ, ξ, η, g) be a Sasakian manifold and μ be a vector field on M which satisfies the next three conditions

$$(2)\quad L_\mu(g) = 0, \qquad [\mu, \xi] = 0, \qquad 1 + \eta(\mu) > 0$$

where L_μ is the Lie differentiation with respect to μ.

New structure tensor fields denoted by $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ are defined by the next equations

$$\tilde{\phi}(X) = \phi(X - \tilde{\eta}(X)\tilde{\xi}), \qquad \tilde{\eta} = (1 + \eta(\mu))^{-1} \cdot \eta, \qquad \tilde{\xi} = \xi + \mu$$

and

$$\tilde{g}(X, Y) = (1 + \eta(\mu))^{-1} \cdot g(X - \tilde{\eta}(X)\tilde{\xi}, Y - \tilde{\eta}(Y)\tilde{\xi}) + \tilde{\eta}(X) \cdot \tilde{\eta}(Y)$$

where X and Y are vector fields on M.

Then $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is also a Sasakian manifold.

$(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is called Sasakian manifold deformed with respect to μ. Let $(S^{2(n+m)-1}, \phi, \xi, \eta, g)$ be the unit sphere with the standard Sasakian structure and be imbedded in the Euclidean space $E^{2(n+m)}$ with coordinates $(x_1, y_1, \cdots\cdots, x_{n+m}, y_{n+m})$. At the point P on $S^{2(n+m)-1}$ we put

$$\xi_P = \sum_{j=1}^{n+m} (x_j \partial y_j - y_j \partial x_j) \quad \text{and} \quad \eta_P = \sum_{j=1}^{n+m} (x_j dy_j - y_j dx_j)$$

where ∂x_j and ∂y_j are vector fields on $E^{2(n+m)}$ usually denoted by $\partial/\partial x_j$ and $\partial/\partial y_j$ respectively and $(x_1, y_1, \cdots\cdots, x_{n+m}, y_{n+m})$ is the coordinates of the point P. Let D be the distribution defined by $\eta = 0$. When we introduce a complex structure J on $E^{2(n+m)}$ as $z_j = x_j + iy_j$ for $j = 1, 2, \cdots\cdots, n+m$, then ϕ is defined as the restriction of J on D which is the orthogonal complement of $R \cdot \xi$ in the tangent space at each point on

$S^{2(n+m)-1}$ and 0 in $R\cdot\xi$. When we put

$$\mu = \sum_{j=1}^{n+m} r_j(x_j \partial y_j - y_j \partial x_j)$$

where $(r_1, r_2, \cdots\cdots, r_{n+m})$ is a $(n+m)$−tuple of real numbers such that

$$1 + \sum_{j=1}^{n+m} r_j((x_j)^2 + (y_j)^2) > 0$$

on $S^{2(n+m)-1}$, then μ satisfies condition (2)

3. Generalized Brieskorn manifolds.

Let C^{n+m} be the complex vector space of $(n+m)$−tuples of complex numbers $z=(z_1, z_2, \cdots\cdots, z_{n+m})$ and a_{kj} be positive integers and α_{kj} be real numbers, $k=1, 2, \cdots\cdots, m$, $j=1, 2, \cdots\cdots, n+m$. Let

(3) $f_k(z_1, z_2, \cdots\cdots, z_{n+m}) = \sum_{j=1}^{n+m} \alpha_{kj} z_j^{a_{kj}}, \quad k=1, 2, \cdots\cdots, m.$

be a collection of complex polynomials. Let $V_k = f_k^{-1}(0)$ and let

$$V = \bigcap_{k=1}^{m} V_k. \quad \text{Let } d_k = \text{L.C.M}(a_{k1}, a_{k2}, \cdots\cdots, a_{k,n+m}), \quad q_{kj} = d_k/a_{kj}.$$

We suppose

a) V is a complete intersection of the V_k.

b) V has an isolated singularity at the origin.

c) q_{kj} is independent of k (let $q_j = q_{kj}$).

Let $B^{2n-1} = V \cap S^{2(n+m)-1} \subset C^{n+m}$. B^{2n-1} is called a generalized Brieskorn manifold [2]. B^{2n-1} is a $(2n-1)$-dimensional submanifold in $S^{2(n+m)-1}$. Denoting by $x_1, y_1, \cdots\cdots, x_{n+m}, y_{n+m}$, the real coordinates of C^{n+m} such that $z_j = x_j + iy_j$ $(j=1, 2, \cdots\cdots, n+m)$, we define a real vector field $\tilde{\xi}$ on C^{n+m} by

$$\tilde{\xi} = \sum_{j=1}^{n+m} A_j(x_j \partial y_j - y_j \partial x_j)$$

where $A_j = Aq_j$ for a positive constant A, $(j=1, 2, \cdots\cdots, n+m)$. This vector field is tangent to $S^{2(n+m)-1}$. Let $\mu = \tilde{\xi} - \xi$, μ satisfies the conditions of the theorem A because of $1+\eta(\mu) = \sum_{j=1}^{n+m} A_j((x_j)^2 + (y_j)^2) > 0$.

We apply the theorem A to $(S^{2(n+m)-1}, \phi, \xi, \eta, g)$ with respect to μ and denote by $(S^{2(n+m)-1}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ the deformed Sasakian sphere with respect to μ. The explicit formulas of $\tilde{\eta}$ and $\tilde{\xi}$ are given by

$$\tilde{\eta} = \sum_{j=1}^{n+m} (K)^{-1}(x_j dy_j - y_j dx_j)$$

and

$$\tilde{\xi} = \sum_{j=1}^{n+m} A_j(x_j \partial y_j - y_j \partial x_j)$$

where $K(x_1, y_1, \cdots\cdots, x_{n+m}, y_{n+m}) = \sum_{j=1}^{n+m} A_j((x_j)^2 + (y_j)^2)$.

We will show that the vector field $\tilde{\xi}$ is tangent to B^{2n-1}. Let

R_k = the real part of $f_k(z_1, z_2, \cdots\cdots, z_{n+m})$

and

I_k = the imaginary part of $f_k(z_1, z_2, \cdots\cdots, z_{n+m})$,

then we see that $\tilde{\xi}R_k = -Ad_kI_k$ and $\tilde{\xi}I_k = Ad_kR_k$, and therfore,

$\tilde{\xi}f_k = \tilde{\xi}R_k + i\tilde{\xi}I_k = Ad_k(-I_k + iR_k) = 0$, for $k=1, 2, \cdots\cdots, m$.

These prove that the vector field $\tilde{\xi}_{|B^{2n-1}}$ is tangent to B^{2n-1}. Let ι ; $B^{2n-1} \to S^{2(n+m)-1}$ be inclusion mapping. We define four tensor fields $(\hat{\phi}, \hat{\xi}, \hat{\eta}, \hat{g})$ on B^{2n-1} by the next equations

(4) $\hat{\phi} = \tilde{\phi}_{|B^{2n-1}}, \qquad \hat{\xi} = \tilde{\xi}_{|B^{2n-1}}, \qquad \hat{\eta} = \iota^* \tilde{\eta} \qquad \text{and} \qquad \hat{g} = \iota^* \tilde{g}.$

We will prove that the restriction $\tilde{\phi}_{|B^{2n-1}}$ is well defined. For any tangent vector X of B^{2n-1} which belongs to D on $S^{2(n+m)-1}$ we get

(5) $\tilde{\phi}(X)(f_k) = (J(X))(f_k) = i \cdot X(f_k) = 0$, for $k = 1, 2, \cdots\cdots, m$.

Since $T_P B^{2n-1} = (D_P + R \cdot \tilde{\xi}_P) \cap T_P B^{2n-1} = (D_P \cap T_P B^{2n-1}) + R \cdot \hat{\xi}_P$ is an orthogonal decomposition with respect to $\hat{g}_P$ where $T_P B^{2n-1}$ is the tangent space at point P of B^{2n-1} and since $\tilde{\phi}(\tilde{\xi}) = \hat{\phi}(\hat{\xi}) = 0$, we find from (5) that $\hat{\phi}$ is a mapping from the tangent space $T_P B^{2n-1}$ to itself. Hence we find that $(\hat{\phi}, \hat{\xi}, \hat{\eta}, \hat{g})$ define a contact metric structure on B^{2n-1}. From B^{2n-1} is an invariant sudmanifold of $(S^{2(n+m)-1}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$, by the theorem of M. Okumura 〔1〕, we have

Theorem. Every generalized Brieskorn manifold admits many Sasakian structures.

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