On Partial Stability of Solutions of a System of Ordinary Differential Equations

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1. Introduction

Explicit solutions of differential equations are often wholly out of the question.

Our purpose in solving most of the differential equations is to see their qualities rather than the concrete expressions of their solutions such as series or elementary functions.

Liapunov's second method enables us to decide stability and boundedness from the differential equations without any knowledge of their solutions. (cf. [2], [3], [4])

By using this method, we obtained several results with respect to stability of solutions of a differential equation, $\frac{dx}{dt} = F(t, x)$, in the previous paper [1].

In many applications, we need to see the qualities not of the whole solution but of the partial.

In this paper, we describe several results concerning the partial stability of the solutions of differential equations.

2. Definitions and Notations

Let I denote the interval $a \leqq t < \infty$ for some real number a or $a = -\infty$ and R^n denote Euclidean n-space. For $x \in R^n$, let $|x|$ be any norm of x.

Let $n > 0$ and $m > 0$ be two integers, and consider two continuous functions $f : I \times D \times R^m \rightarrow R^n$, $g : I \times D \times R^m \rightarrow R^m$, where D is a domain (i.e. an open connected set) of R^n, containing the origin. We assume that $f(t,0,0) = 0$ and $g(t,0,0) = 0$ for every $t \in I$ and further that f and g are smooth enough in order that, through every point of $I \times D \times R^m$, there passes one and only one solution of the differential system

$$x' = f(t,x,y), \qquad y' = g(t,x,y). \qquad (1)$$

To shorten our notation, we shall write z for the vector $(x,y) \in R^{n+m}$ and also $z(t,t_0,z_0) = (x(t,t_0,z_0), y(t,t_0,z_0))$ for the solution of (1) starting from z_0 at t_0.

For the right maximal interval where $z(t,t_0,z_0)$ is defined, we write $J^+(t_0,z_0)$ or simply J^+. We shall denote by CIP the family of all continuous increasing, positive definite functions. For such a function, we shall usually write $a \in$ CIP.

We introduce the following definitions.

【Definition 1】 The solution $z = 0$ of (1) is partially stable with respct to x if for any $\varepsilon > 0$ and any $t_0 \in I$, there exists a $\delta(t_0, \varepsilon) > 0$ such that $|z_0| < \delta$ implies $|x(t,t_0,z_0)| < \varepsilon$ for all $t \in J^+$.

【Definition 2】 The solution $z = 0$ of (1) is partially asymptotically stable with respect to x if it is partially stable with respect to x and if there exists a $\delta_0(t_0) > 0$ such that $|z_0| < \delta_0(t_0)$ implies $x(t,t_0,z_0) \rightarrow 0$ as $t \rightarrow \infty$.

【Definition 3】 For a continuous function $V : I \times D \times R^m \rightarrow R$, we define the function

$$V'_{(1)}(t,x,y)=\overline{\lim_{h\to+0}}\frac{1}{h}\{V(t+h,\ x+hf(t,x,y),\ y+hg(t,x,y))-V(t,x,y)\}.$$

In case $V(t,x,y)$ has continuous partial derivatives of the first order, it is evident that

$$V'_{(1)}(t,x,y)=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}\cdot f(t,x,y)+\frac{\partial V}{\partial y}\cdot g(t,x,y),$$

where "$\cdot$" denote a scalar product.

3. Preliminary Results

【Theorem 1】 *If there exists a continuous function* $V: I\times D\times R^m\to R$ *such that, for some* $a\in CIP$ *and every* $(t,x,y)\in I\times D\times R^m$,

(i) $V(t,0,0)=0$

(ii) $a(|x|)\leqslant V(t,x,y)$,

(iii) $V'_{(1)}(t,x,y)\leqslant 0$,

then the solution $z=0$ *of* (1) *is partially stable with respect to* x.

For the proof, see [6].

【Theorem 2】 *If there exists a continuous function* $V: I\times D\times R^m\to R$ *such that, for some functions* a, $c\in CIP$ *and every* $(t,x,y)\in I\times D\times R^m$,

(i) $V(t,0,0)=0$,

(ii) $a(|x|)\leqslant V(t,x,y)$,

(iii) $V'_{(1)}(t,x,y)\leqslant -c(V(t,x,y))$,

then the solution $z=0$ *of* (1) *is partially asymptotically stable with respect to* x.

For the proof, see [5].

Let $F(t,x)$ be a continuous function $F: I\times D\to R^n$. We assume that $F(t,0)=0$. For a system of differential equations

$$x'=F(t,x), \qquad (2)$$

we have the following theorems.

【Theorem 3】 *If there exists a continuous function* $V: I\times D\to R$ *such that,*

(i) $V(t,0)=0$

(ii) $a(t,|x|)\leqslant V(t,x)$, *where the continuous function* $a: I\times R^+\to R^+$ $(R^+=[0,\infty))$ *is* $a(t,0)=0$, $a(t,r)>0$ for $r\neq 0$ *and increases monotonically with respect to* t *for each fixed* r,

(iii) $V'_{(2)}(t,x)\leqslant 0$,

then the solution $x=0$ *of* (2) *is stable.*

For the proof, see [1].

【Theorem 4】 *If there exists a continuous function* $V: I\times D\to R$ *such that,*

(i) $V(t,0)=0$,

(ii) $a(t,|x|)\leqslant V(t,x)$, *where the continuous function* $a: I\times R^+\to R^+$ *is* $a(t,0)=0$, $a(t,r)>0$ *for* $r\neq 0$ *and increases monotonically with respect to* t *for each fixed* r,

(iii) $V'_{(2)}(t,x)\leqslant -c(t,|x|)$, *where the continuous function* $c: I\times R^+\to R^+$ is $c(t,0)=0$ *and increases* $c(t,r)>0$ *for* $r\neq 0$, *and if* $F(t,x)$ *is bounded when* x *is contained in a compact set, then the solution* $x=0$ *of* (2) *is asymptotically stable.*

For the proof, see [7].

4. Main Results

【Theorem 5】 *If there exists a continuous function* $V: I \times D \times R^m \to R$ *such that,*

(i) $V(t,0,0)=0$,

(ii) $a(t,|x|) \leqq V(t,x,y)$, *where the continuous function* $a: I \times R^+ \to R^+$ *is* $a(t,0)=0$, $a(t,r)>0$ *for* $r \neq 0$ *and increases monotonically with respect to* t *for each fixed* r,

(iii) $V'_{(1)}(t,x,y) \leqq 0$,

then the solution $z=0$ *of* (1) *is partially stable with respect to* x.

[Proof] Corresponding to any $\varepsilon>0$ and any $t_0 \in I$, we have $a(t,\varepsilon) \leqq V(t,x,y)$ for $t \in I$, x such that $|x|=\varepsilon$ and $y \in R^m$. For a fixed $t_0 \in I$, we can choose a $\delta(t_0,\varepsilon)$ such that $|x_0|+|y_0|<\delta$ implies $V(t_0,x_0,y_0)<a(t_0,\varepsilon)$, where $\delta<\varepsilon$, because $V(t,0,0)=0$ and $V(t,x,y)$ is continuous.

Suppose that there exists some t_1 such that $|x_0|+|y_0|<\delta$ implies $|x(t_1,t_0,z_0)|=\varepsilon$, $t_1>t_0$. From the conditions (ii) and (iii), it follows that $V(t_1,x(t_1,t_0,z_0),y(t_1,t_0,z_0)) \leqq V(t_0,x_0,y_0)$ and hence we have

$$\begin{aligned} a(t_1,\varepsilon) &= a(t_1,|x(t_1,t_0,z_0)|) \\ &\leqq V(t_1,x(t_1,t_0,z_0),y(t_1,t_0,z_0)) \\ &\leqq V(t_0,x_0,y_0) \\ &< a(t_0,\varepsilon). \end{aligned}$$

This contradicts the condition (ii). Therefore if $|x_0|+|y_0|<\delta$, then $|x(t,t_0,z_0)|<\varepsilon$ for all $t \geqq t_0$, that is, the solution $z=0$ of (1) is partially stable with respect to x.

【Theorem 6】 *If there exists a continuous function* $V: I \times D \times R^m \to R$ *such that,*

(i) $V(t,0,0)=0$,

(ii) $a(t,|x|) \leqq V(t,x,y)$, *where the continuous function* $a: I \times R^+ \to R^+$ *is* $a(t,0)=0$, $a(t,r)>0$ *for* $r \neq 0$ *and increases monotonically with respect to* t *for each fixed* r,

(iii) $V'_{(1)}(t,x,y) \leqq -c(t,|x|)$, *where the continuous function* $c: I \times R^+ \to R^+$ *is* $c(t,0)=0$ *and* $c(t,r)>0$ *for* $r \neq 0$, *and if* $f(t,x,y)$ *is bounded when* x *is contained a compact set, then the solution* $z=0$ *of* (1) *is partially asymptotically stable with respect to* x.

[Proof] By Theorem 5, the solution of (1) is partially stable with respect to x. Therefore for every $t_0 \in I$, there is a $\delta_0(t_0)>0$ such that $|x_0|<\delta_0(t_0)$ implies $|x(t,t_0,z_0)| \leqq H$, $H>0$.

Suppose that $x(t,t_0,z_0)$ does not tend to zero as $t \to \infty$. Then for some $\varepsilon>0$, there exists a divergent sequence $\{t_k\}$ for which $|x(t_k,t_0,z_0)| \geqq \varepsilon$, where $t_0 \in I$ and $|z_0|<\delta_0(t_0)$. Since $f(t,x,y)$ is bounded for x such that $|x| \leqq H$, there exists a $K>0$ such that $\left|\frac{d}{dt}|x(t,t_0,z_0)|\right|<K$. Therefore, on intervals

$$t_k-\frac{\varepsilon}{2K} \leqq t \leqq t_k+\frac{\varepsilon}{2K}, \tag{3}$$

we have $|x(t,t_0,z_0)| \geqq \frac{\varepsilon}{2}$. We can assume that these intervals are disjoint and $t_1-\frac{\varepsilon}{2K}>t_0$ by taking, if necessary, a subsequence of $\{t_k\}$.

Since $V'_{(1)}(t,x,y) \leqq -c(t,|x|)$, there exists a constant $\gamma>0$ such that $V'_{(1)}(t,x(t,t_0,z_0),y(t,t_0,z_0)) \leqq -\gamma$ on intervals (3), and $V'_{(1)}(t,x(t,t_0,z_0),y(t,t_0,z_0)) \leqq 0$ elsewhere, because $\frac{\varepsilon}{2} \leqq |x(t,t_0,z_0)| \leqq H$ on intervals (3). Therefore $V(t_k+\frac{\varepsilon}{2K},x(t_k+\frac{\varepsilon}{2K},t_0,z_0),y(t_k+\frac{\varepsilon}{2K},t_0,z_0))-V(t_0,x_0,y_0) \leqq -\gamma \cdot \frac{\varepsilon}{2K}k \to -\infty$ as $k \to \infty$, which contradicts $V(t,x,y) \geqq 0$. Thus we see that the solution $z=0$ of (1) is partially asymptotically stable with respect to x.

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