On Partial Boundedness of Solutions of a System of Ordinary Differential Equations

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1. Introduction

Explicit solutions of differential equations are often wholly out of the question.

Our purpose in solving most of the differential equations is to see their qualities rather than the concrete expressions of their solutions such as series or elementary functions.

Liapunov's second method enables us to decide stability and boundedness from the differential equations without any knowledge of their solutions. (cf. [2], [3], [4])

By using this method, we obtained several results with respect to boundedness of solutions of a differential equation, $\frac{dx}{dt} = F(t,x)$, in the previous paper [1].

In many applications, we need to see the qualities not of the whole solution but of the partial.

In this paper, we describe several results concerning the partial boundedness of the solutions of differential equations.

2. Definitions and Notations

Let I denote the interval $0 \leqq t < \infty$ and R^n denote Euclidean n-space.

Let $\|x\|$ denote the norm of x. Let $z = (x, y) \in R^n \times R^m$.

We shall denote by $C(I \times R^n \times R^m, R^k)$ the set of all continuous function f defined on $I \times R^n \times R^m$ with values in R^k.

Let $F(t,x) \in C(I \times D, R^n)$, where D is an open set in R^n. For a system

$$\frac{dx}{dt} = F(t,x), \tag{1}$$

a solution through a point $(t_0, x_0) \in I \times R^n$ will be denoted by such a form as $x(t, t_0, x_0)$.

Let $f(t,x,y) \in C(I \times R^n \times R^m, R^n)$ and $g(t,x,y) \in C(I \times R^n \times R^m, R^m)$.

We consider a system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = f(t,x,y) \\ \dfrac{dy}{dt} = g(t,x,y). \end{cases} \tag{2}$$

Throughout this paper a solution of (2) through a point $(t_0, z_0) = (t_0, x_0, y_0)$ in $I \times R^n \times R^m$ will be denoted by such a form as $(x(t,t_0,z_0),\ y(t,t_0,z_0))$.

We introduce the following definitions.

【Definition 1】 The solutions of the system (1) are equi-bounded, if for any $\alpha > 0$ and $t_0 \in I$, there exists a $\beta(t_0, \alpha) > 0$ such that if $\|x_0\| \leqq \alpha$, $\|x(t, t_0, x_0)\| < \beta(t_0, \alpha)$ for all $t \geqq t_0$.

【Definition 2】 The solutions of the system (1) are uniform-bounded, if the β in the above Definition 1 is independent of t_0.

【Definition 3】 The solutions of the system (1) are equi-ultimately bounded for bound B, if there exists a $B>0$ and if corresponding to any $\alpha>0$ and $t_0 \in I$, there exists a $T(t_0,\alpha)>0$ such $\|x_0\| \leqq \alpha$ implies that $\|x(t,t_0,x_0)\|<B$ for all $t \geqq t_0+T(t_0,\alpha)$.

【Definition 4】 The solutions of the system (2) are partially equi-bounded with respect to x, if for any $\alpha>0$ and any $t_0 \in I$, there exists a $\beta(t_0,\alpha)>0$ such that $\|z_0\| \leqq \alpha$ implies that $\|x(t,t_0,z_0)\|<\beta(t_0,\alpha)$ for all $t \geqq t_0$.

【Definition 5】 The solutions of the system (2) are partially uniform-bounded with respect to x, if the β in the above Definition 4 is independent of t_0.

【Definition 6】 The solution of (2) are partially equi-ultimately bounded for bound B with respect to x, if there exists a $B>0$ and if corresponding to any $\alpha>0$ and $t_0 \in I$, there exists a $T(t_0,\alpha)>0$ such that $\|z_0\| \leqq \alpha$ implies that $\|x(t,t_0,z_0)\|<B$ for all $t \geqq t_0+T(t_0,\alpha)$.

【Definition 7】 Let $V(t,x)$ and $V(t,x,y)$ be continuous scalar functions defined on open sets, and which satisfy locally a Lipschitz condition with respect to x and (x,y) respectively. Corresponding to $V(t,x)$ and $V(t,x,y)$, we define the functions

$$V'_{(1)}(t,x)=\overline{\lim_{h\to+0}}\frac{1}{h}\{V(t+h,x+hF(t,x))-V(t,x)\}$$

and

$$V'_{(2)}(t,x,y)=\overline{\lim_{h\to+0}}\frac{1}{h}\{V(t+h,x+hf(t,x,y),y+hg(t,x,y))-V(t,x,y)\}$$

respectively. In case $V(t,x)$ and $V(t,x,y)$ have continuous partial derivatives of the first order, it is evident that

$$V'_{(1)}(t,x)=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}\cdot F(t,x)$$

and

$$V'_{(2)}(t,x,y)=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}\cdot f(t,x,y)+\frac{\partial V}{\partial y}\cdot g(t,x,y)$$

where "$\cdot$" denotes a scalar product.

3. Preliminary Results

【Theorem 1】 *Suppose that there exists a function* $V(t,x) \in C(I\times R^n,R)$, *which satisfies the following conditions*;

(i) $a(t,\|x\|) \leqq V(t,x)$, *where the function* $a(t,r)$ *is continuous in* (t,r) *and* $a(t,r)\to\infty$ *uniformly in* t *as* $r\to\infty$,

(ii) $V'_{(1)}(t,x) \leqq 0$.

Then the solutions of the system (1) *are equi-bounded.*

【Theorem 2】 *Suppose that there exists a function* $V(t,x) \in C(I\times S,R)$, *where* $S=\{x\mid \|x\| \geqq K\}$ *for a sufficiently large* K, *which satisfies the following conditions*;

(i) $a(t,\|x\|) \leqq V(t,x) \leqq b(\|x\|)$, *where the function* $a(t,r)$ *is continuous in* (t,r) *and* $a(t,r)\to\infty$ *uniformly in* t *as* $r\to\infty$, *and the function* $b(r)$ *is continuous,*

(ii) $V'_{(1)}(t,x) \leqq 0$.

Then the solutions of the system (1) *are uniform-bounded.*

【Theorem 3】 *Suppose that there exists a function* $V(t,x) \in C(I\times R^n,R)$, *which satisfies the following conditions*;

(i) $a(t,\|x\|) \leqq V(t,x)$ *for* $\|x\| \geqq B$, *where the function* $a(t,r)$ *is continuous in* (t,r) *and monotone-increasing with respect to* t *for each fixed* r *and to* r *for each fixed* t, *and* $a(t,r)\to\infty$ *uniformly in* t *as* $r \to \infty$,

(ii) $V'_{(1)}(t,x) \leqq -cV(t,x)$, *where* $c > 0$ *is a constant.*

Then the solutions of the system (1) *are equi-ultimately bounded for bound* B.

For proof of these theorems, see reference [1].

4. Main Results

【Theorem 4】 *Suppose that there exists a function* $V(t,x,y) \in C(I \times R^n \times R^m, R)$ *which satisfies the following conditions*;

(i) $a(t, \|x\|) \leqq V(t,x,y)$, *where the function* $a(t,r)$ *is continuous in* (t,r) *and* $a(t,r) \to \infty$ *uniformly in* t *as* $r \to \infty$,

(ii) $V'_{(2)}(t,x,y) \leqq 0$.

Then the solutions of the system (2) *are partially equi-bounded with respect to x.*

Proof. For any given $\alpha > 0$, let $(x(t,t_0,z_0), y(t,t_0,z_0))$ be a solution of (2) such that $t_0 \in I$ and $\|z_0\| \leqq \alpha$. Since $V(t,x,y)$ is continuous, there exists a $K(t_0,\alpha) > 0$ such that if $\|z_0\| \leqq \alpha$, $V(t_0,x_0,y_0) \leqq K(t_0,\alpha)$. By (i), we can choose a $\beta(t_0,\alpha) > 0$ so large that $K(t_0,\alpha) < a(t,\beta(t_0,\alpha))$ for any $t \geqq t_0$. Suppose that $\|x(t_1,t_0,z_0)\| = \beta(t_0,\alpha)$ at some t_1, $t_1 \geqq t_0$. By (i) and (ii), we have

$$K(t_0,\alpha) < a(t_1, x(t_1,t_0,z_0)) \leqq V(t_1, x(t_1,t_0,z_0), y(t_1,t_0,z_0))$$
$$\leqq V(t_0,x_0,y_0) \leqq K(t_0,\alpha).$$

This is a contradiction. Thus there exists a $\beta(t_0,\alpha) > 0$ such that $\|z_0\| \leqq \alpha$ implies $\|x(t,t_0,z_0)\| < \beta(t_0,\alpha)$ for all $t \geqq t_0$. This proves that the solutions of the system (2) are partially equi-bounded with respect to x.

【Theorem 5】 *Suppose that there exists a function* $V(t,x,y) \in C(I \times D_K, R)$, *where* $D_K = \{(x,y) \mid \|x\| + \|y\| \geqq K, x \in R^n, y \in R^m\}$ *for a sufficiently large* K, *which satisfies the following conditions*;

(i) $a(t, \|x\|) \leqq V(t,x) \leqq b(\|x\| + \|y\|)$, *where the function* $a(t,r)$ *is continuous in* (t,r) *and* $a(t,r) \to \infty$ *uniformly in* t *as* $r \to \infty$, *and the function* $b(r)$ *is continuous,*

(ii) $V'_{(2)}(t,x,y) \leqq 0$.

Then the solutions of the system (2) *are partially uniform-bounded with respect to x.*

Proof. For any given $\alpha > 0$, by continuity of the function $b(r)$, there exists a $B(\alpha) > 0$ such that $B(\alpha) < a(t,\beta(\alpha))$ for any $t \geqq t_0$.
$b(\|x\| + \|y\|) \leqq B(\alpha)$ for $K \leqq \|x\| + \|y\| \leqq \alpha$, and we can choose a $\beta(\alpha) > 0$ so large that $B(\alpha) < a(t,\beta(\alpha))$ for any $t \geqq t_0$.

Suppose that $\|x(t,t_0,z_0)\| = \beta(\alpha)$ at some t. Then there exist t_1 and t_2, $t_0 \leqq t_1 \leqq t_2$, such that

$$\|x(t_1,t_0,z_0)\| + \|y(t_1,t_0,z_0)\| = \alpha$$

and

$$\|x(t_2,t_0,z_0)\| = \beta(\alpha).$$

By (i) and (ii), we have

$$a(t_2,\beta(\alpha)) = a(t_2, \|x(t_2,t_0,z_0)\|) \leqq V(t_2, x(t_2,t_0,z_0), y(t_2,t_0,z_0))$$
$$\leqq V(t_1, x(t_1,t_0,z_0), y(t_1,t_0,z_0))$$
$$\leqq b(\|x(t_1,t_0,z_0)\| + \|y(t_1,t_0,z_0)\|) = b(\alpha) \leqq B(\alpha).$$

This contracdicts the choice of $\beta(\alpha)$. Thus $\|x(t,t_0,z_0)\| < \beta(\alpha)$ for all $t \geqq t_0$. This shows the partial uniform-boundedness with respect to x of solutions of the system (2).

【Theorem 6】 *Suppose that there exists a function* $V(t,x,y) \in C(I \times R^n \times R^m, R)$ *which satisfies the following conditions*;

(i) $a(t, \|x\|) \leqq V(t,x,y)$ *for* $\|x\| \geqq B$, *where the function* $a(t,r)$ *is continuous in* (t,r) *and monotone-increasing with respect to* t *for each fixed* r *and to* r *for each fixed* t, *and* $a(t,r) \to \infty$ *uniformly in* t *as* $r \to \infty$,

(ii) $V'_{(2)}(t,x,y) \leqq -cV(t,x,y)$, *where* $c > 0$ *is a constant.*

Then the solutions of the system (2) *are partially equi-ultimately bounded for bound* B *with respect to* x.

Proof. For any given $\alpha > 0$ and $t_0 \in I$, by the continuity of $V(t,x,y)$, there exists a $K(t_0,\alpha) > 0$ such that if $\|z_0\| \leqq \alpha$, $V(t_0,x_0,y_0) \leqq K(t_0,\alpha)$.

Let $(x(t,t_0,z_0),y(t,t_0,z_0))$ be a solution of (2) such that $\|z_0\| \leqq \alpha$. It is bounded for all $t \geqq t_0$. Suppose that there exists some

$t_1 > t_0 + \frac{1}{c}\log\frac{K(t_0,\alpha)}{a(t_0,B)}$ such that $\|x(t_1,t_0,z_0)\| \geqq B$. From (i) and (ii),

$$\begin{aligned} a(t_0,B) &\leqq a(t_1, \|x(t_1,t_0,z_0)\|) \leqq V(t_1,x(t_1,t_0,z_0),y(t_1,t_0,z_0)) \\ &\leqq V(t_0,x_0,y_0)\exp\{-c(t_1-t_0)\} \\ &< K(t_0,\alpha)\exp\left\{-\log\frac{K(t_0,\alpha)}{a(t_0,B)}\right\} = a(t_0,B). \end{aligned}$$

This is a contradiction. Therefore, if $t > t_0 + \frac{1}{c}\log\frac{K(t_0,\alpha)}{a(t_0,B)}$, we have $\|x(t,t_0,z_0)\| < B$. Thus the solutions of the system (2) are partially equi-ultimately bounded for bound B with respect to x.

References

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