On the Existence of the Almost Periodic Solutions of the Almost Periodic System (II)

Shoichi Seino
Miki Kudo
Masamichi Aso

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1. Introduction

It is well-known that the boundedness property of solutions of the periodic system implies the existence of a periodic solution. However, for an almost periodic equation, the boundedness of solutions does not necessarily imply the existence of an almost periodic solutions even for scalar equations.

Z. Opial 〔1〕 has constructed an equation with all of its solutions bounded but not almost periodic. A.M.Fink and P.O.Frederickson 〔2〕, by using Opial's equation, have constructed an almost periodic equation which has no almost periodic solutions, but the solutions are uniformly ultimated bounded. Thus, in discussing the existence of an almost periodic solution, several kind stability properties of a bounded solution were assumed. For example, Miller assumed that the bounded solution is totally stable, and Seifert assumed the Σ-stability of the bounded solution, while Sell assumed the stability under disturbances from the hull. They assumed that the solutions are unique, but these results can be obtained by using the property of asymptotically almost periodic functions without the uniqueness of solutions 〔3〕, 〔4〕.

In 〔10〕, we proved the existence theorem of an almost periodic solution by using a generalized Liapunov function, following an idea of T. Yoshizawa 〔5〕.

In this paper, we shall show, for an almost periodic system, a refined result of our theorem by assuming that solutions of the system of differential equations are ultimately bounded.

2. Definitions and Notations

We shall use the following notations. Let R denote the whole real line, I denote the interval $0 \leqq t < \infty$ and R^n denote Euclidean n-space. For $x \in R^n$, let $|x|$ be the Euclidean norm of x and S_{B^*} be a domain such that $|x| < B^*$, $B^* > 0$. We shall denote by $C(J \times D, R^n)$ the set of all continuous functions defined on $I \times S_{B^*}$ with values in R^n, where J is a subset of R and D is a subset of R^n.

We consider an almost periodic system

$$x' = f(t, x), \qquad (1)$$

where $f(t, x) \in C(R \times R^n, R^n)$ and $f(t, x)$ is almost periodic in t uniformly for $x \in R^n$.
In studying the behavior of a pair of solutions, it is natural to introduce the product system

$$x' = f(t, x), \quad y' = f(t, y). \qquad (2)$$

Throughout this paper a solution through a point (t_0, x_0) will be denoted by such a form as $x(t, t_0, x_0)$.

We introduce the following definitions.

〔Definition 1〕. A solution $x(t, t_0, x_0)$ of (1) is bounded, if there exists a $\beta > 0$ such that $|x(t, t_0, x_0)| < \beta$ for all $t \geqq t_0$, where β may depend on each solution.

〔Definition 2〕. The solutions of (1) are ultimately bounded for bound B, if there exist a $B > 0$ and a $T > 0$ such that for every solution $x(t, t_0, x_0)$ of (1), $|x(t, t_0, x_0)| < B$ for all $t \geqq t_0 + T$, where B is independent of the particular solution while T may depend on each solution.

〔Definition 3〕. The solution $\varphi(t)$ of (1) is stable, if for any $\varepsilon > 0$ and any $t_0 \in I$, there exists a $\delta(t_0, \varepsilon) > 0$ such that $|x_0 - \varphi(t_0)| < \delta(t_0, \varepsilon)$ implies $|x(t, t_0, x_0) - \varphi(t)| < \varepsilon$ for all $t \geqq t_0$.

〔Definition 4〕. The solution $\varphi(t)$ of (1) is quasi-equiasymptotically stable in the large, if for any $\alpha > 0$, any $\varepsilon > 0$ and any $t_0 \in I$, there exists a $T(t_0, \varepsilon, \alpha) > 0$ such that if $|x_0 - \varphi(t_0)| \leqq \alpha$, then $|x(t, t_0, x_0) - \varphi(t)| < \varepsilon$ for all $t \geqq t_0 + T(t_0, \varepsilon, \alpha)$.

〔Definition 5〕. The solution $\varphi(t)$ of (1) is equi-asymptotically stable in the large, if it is stable and is quasi-equiasymptotically stable in the large.

〔Definition 6〕. Corresponding to a continuous scalar function $V(t, x, y)$ defined on an open set in $R \times D \times D$, we define the function

$$V'_{(2)}(t, x, y) = \overline{\lim_{h \to 0+}} \frac{1}{h} \left\{ V(t + h, x + hf(t, x), y + hf(t, y)) - V(t, x, y) \right\}.$$

3. Preliminary Result

【Theorem 1】

Suppose that there exists a Liapunov function $V(t, x, y)$ *defined on* $I \times S_{B^*} \times S_{B^*}$ *which satisfies the following conditions;*

(i) $a(t, |x-y|) \leqq V(t, x, y) \leqq b(t, |x-y|)$, *where the function* $a(t, r)$ *is continuous in* (t, r), $a(t, 0) \equiv 0$, $a(t, r) > 0$ *for any* $r \neq 0$, *and increases monotonically with respect to* t *and* r, *and* $b(t, r)$ *is a continuous function in* (t, r) *and increases monotonically with respect to* r,

(ii) $|V(t, x_1, y_1) - V(t, x_2, y_2)| \leqq K(|x_1 - x_2| + |y_1 - y_2|)$, *where* $K > 0$ *is a constant,*

(iii) $V'_{(2)}(t, x, y) \leqq -cV(t, x, y)$, *where* $c > 0$ *is a constant.*

Moreover, suppose that there exists a solution $p(t)$ *of* (1) *such that* $|p(t)| \leqq B < B^*$ *for all* $t \geqq 0$. *Then, in the region* $R \times S_B{}^*$, *there exists a unique equi-asymptotically stable almost periodic solution* $q(t)$ *of* (1) *which is bounded by* B.

In particular, if $f(t, x)$ *is periodic in* t *of period* ω, *then there exists a unique equi-asymptotically stable periodic solution of* (1) *of period* ω.

For proof, see 〔10〕.

4. Result

【Theorem 2】

Suppose that the solutions of (1) *are ultimately bounded for bound* $B > 0$ *and that there exists a Liapunov function* $V(t, x, y)$ *defined on* $I \times S_{B^*} \times S_{B^*}$, $B < B^*$, *which satisfies the following conditions;*

(i) $a(t, |x-y|) \leqq V(t, x, y) \leqq b(t, |x-y|)$, *where the function* $a(t, r)$ *is continuous* *in* (t, r), $a(t,0) \equiv 0$, $a(t,r) > 0$ *for any* $r \neq 0$, *and increases monotonically with respect to* t *and* r, *and* $b(t,r)$ *is a continuous function*, $b(t,0) = 0$ *and increases monotonically with respect to* r,

(ii) $|V(t,x_1,y_1) - V(t,x_2,y_2)| \leqq K(|x_1-x_2| + |y_1-y_2|)$, *where* $K>0$ *is a constant*,

(iii) $V'_{(2)}(t,x,y) \leqq -cV(t,x,y)$, *where* $c>0$ *is a constant*.

Then system (1) *has a unique almost periodic solution which is equi-asymptotically stable in the large.*

Proof. Since the solutions of (1) are ultimately bounded for bound B, the each solution of (1) is bounded. Therefore system (1) has a unique almost periodic solution by Theorem 1.

Next we shall prove that this almost periodic solution is equi-asymptotically stable in the large. Let $x(t,t_0,x_0)$ and $y(t,t_0,y_0)$ be any two solutions of (1).

Since the solutions of (1) are ultimately bounded for bound B, there exist $T_1>0$ and $T_2>0$ such that $|x(t,t_0,x_0)|<B$ for all $t \geqq t_0+T_1$ and $|y(t,t_0,y_0)|<B$ for all $t \geqq t_0+T_2$. Let $T_3 = \max(T_1,T_2)$. Then we have that $|x(t,t_0,x_0)|<B$ and $|y(t,t_0,y_0)|<B$ for all $t \geqq t_0+T_3$.

We shall show that for given any positive number δ, if we have at some $t=T$ $(\geqq t_0+T_3)$

$$|x(T,t_0,x_0) - y(T,t_0,y_0)| \geqq \delta,$$

then we have at some T' $(>T)$

$$|x(T',t_0,x_0) - y(T',t_0,y_0)| < \delta. \qquad (3)$$

Let Δ_1 and Δ_2 be two domains such as

$$T \leqq t < \infty,\quad |x| \leqq B,\quad |y| \leqq B$$

and

$$T \leqq t < \infty,\quad |x-y| < \delta'$$

respectively, where $\delta' < \delta$.

If we consider a function

$$W(t,x,y) = e^{\beta t}V(t,x,y) \qquad (c>\beta>0)$$

in $\Delta_1 - \Delta_2$, then we have

1^0 $W(t,x,y) > 0$, since $|x-y| > 0$,

2^0 $W(t,x,y)$ tends to infinity uniformly for (x,y) as $t \to \infty$.

And it is clear that $W(t,x,y)$ satisfies locally for t the Lipschitz condition with regard to (x,y). Moreover we have

$$W'_{(2)}(t,x,y) = \overline{\lim_{h\to 0+}} \frac{1}{h}\Big(W(t+h,x+hf(t,x),y+hf(t,y)) - W(t,x,y)\Big)$$

$$= \overline{\lim_{h\to 0+}} \frac{1}{h}\Big(e^{\beta(t+h)}V(t+h,x+hf(t,x),y+hf(t,y)) - e^{\beta t}V(t,x,y)\Big)$$

$$= \overline{\lim_{h\to 0+}} \frac{1}{h}\Big\{e^{\beta(t+h)}\Big(V(t+h,x+hf(t,x),y+hf(t,y)) - V(t,x,y)\Big) + (e^{\beta(t+h)} - e^{\beta t})V(t,x,y)\Big\}$$

$$= e^{\beta t}\overline{\lim_{h\to 0+}} \frac{1}{h}\Big(V(t+h,x+hf(t,x),y+hf(t,y)) - V(t,x,y)\Big) + \beta e^{\beta t}V(t,x,y)$$

$$= e^{\beta t}V'_{(2)}(t,x,y) + \beta e^{\beta t}V(t,x,y)$$

$$\leqq e^{\beta t}(-c+\beta)V(t,x,y) < 0$$

Therefore

3^0 $W(t,x,y)$ satisfies locally the Lipschitz condition with regard to (x,y) and for all points in the interior of $\Delta_1-\Delta_2$ we have $W'_{(2)}(t,x,y)<0$.

Now suppose that the assertion (3) is not true for δ.

Let Δ_3 and Δ_4 be two domains such as

$$|x|\leqq B,\quad |y|\leqq B$$

and $\quad |x-y|<\delta \quad$ respectively .

Then we can choose T^* by 2^0 such that

$$\min_{\Delta_3-\Delta_4} e^{\beta T^*}V(T^*,x,y) > \max_{\Delta_3-\Delta_4} e^{\beta T}V(T,x,y)\ ,$$

while by 3^0

$$W(T^*,x(T^*,t_0,x_0),\ y(T^*,t_0,y_0)) < W(T,x(T,t_0,x_0),\ y(T,t_0,y_0))$$

This is a contradiction. Therefore the assertion (3) is true.

Hence we can assume without loss of generality that

$$|x(T,t_0,x_0)-y(T,t_0,y_0)|<\delta.$$

For a given $\varepsilon>0$, let δ_0 be $\inf\{V(t,x,y)\mid |x-y|=\varepsilon,\ t_0\leqq t\}$.

Then since $V(t,x,y)$ is positive for $|x-y|>0$, it is clear that $\delta_0>0$, and δ_0 is independent of t.

Now we put $\delta=\min(\delta_0/_K,\ \varepsilon)$ as above δ.

Then the inequality $|x(t,t_0,x_0)-y(t,t_0,y_0)|<\varepsilon$ (4)

holds for $t>T$: Namely otherwise suppose that we have at some $t=t'$

$$|x(t',t_0,x_0)-y(t',t_0,y_0)|=\varepsilon\ .$$

Now consider the function $V(t,x(t,t_0,x_0),y(t,t_0,y_0))$ for $T\leqq t\leqq t'$

and then $\quad V(t',x(t',t_0,x_0),y(t',t_0,y_0))\geqq\delta_0$, hence we have

$$V(T,x(T,t_0,x_0),y(T,t_0,y_0))\geqq\delta_0 \qquad (5)$$

since this function is a nonincreasing function of t by condition (iii) .

Moreover we have

$$\begin{aligned} &V(T,x(T,t_0,x_0),y(T,t_0,y_0)) \\ &\quad =V(T,x(T,t_0,x_0),y(T,t_0,y_0))-V(T,y(T,t_0,y_0),y(T,t_0,y_0)) \\ &\quad \leqq K\,|x(T,t_0,x_0)-y(T,t_0,y_0)| \\ &\quad <K\delta\leqq\delta_0\ . \end{aligned}$$

Hence we have

$$V(T,x(T,t_0,x_0),y(T,t_0,y_0))<\delta_0\ .$$

This contradicts (5) and hence (4) holds good.

Since $x(t,t_0,x_0)$ and $y(t,t_0,y_0)$ are any solutions of (1), we have that an almost periodic solution of (1) is equi-asymptotically stable in the large as follows.

Let $p(t)$ be an almost periodic solution of (1).

Suppose that there is no T which is in the definition of equi-asymptotically stable in the large. Then there exists some $\eta>0$, $t_0\geqq 0$ and sequence $\{x_k\}$,$\{t_k\}$ such that $|x_k-p(t_0)|\leqq\alpha$ (α is arbitrary), $t_k\to\infty$ as $k\to\infty$ and

$$|x(t_k,t_0,x_k)-p(t_k)|\geqq\eta\ . \qquad (6)$$

On any compact interval $[t_0,t_0+N]$, the sequence $\{x(t,t_0,x_k)\}$ is uniformly bounded and equicontinuous, and hence we can find a solution $x(t,t_0,\bar{x})$ of (1) defined for all $t\geqq t_0$, where $|p(t_0)-\bar{x}|\leqq\delta_0$, and moreover, a subsequence of $\{x(t,t_0,x_k)\}$ tends to $x(t,t_0,\bar{x})$ uniformly on any compact interval .

Since every solution tends to $p(t)$ as $t\to\infty$, for any δ'' there exists some t_1 such that

$$|x(t_1,t_0,\overline{x}) - p(t_1)| < \frac{1}{2}\delta'' . \qquad (7)$$

Denoting by $\{x(t,t_0,x_k)\}$ the subsequence again, if k is sufficiently large, we have

$$|x(t_1,t_0,x_k) - x(t_1,t_0,\overline{x})| < \frac{1}{2}\delta''.$$

From this and (7) it follows that

$$|x(t_1,t_0,x_k) - p(t_1)| < \delta''$$

Hence for any $t > t_1 + \frac{1}{c}\log\frac{b(t_1,\delta'')}{a(t_0,\eta)}$ we have

$$a(t, |x(t,t_0,x_k) - p(t)|) \leqq V(t, x(t,t_0,x_k), p(t))$$

$$\leqq V(t_1, x(t_1,t_0,x_k), p(t_1))\, e^{-c(t-t_1)}$$

$$\leqq b(t_1, |x(t_1,t_0,x_k) - p(t_1)|)\, e^{-c(t-t_1)}$$

$$\leqq b(t_1,\delta'')\, \frac{a(t_0,\eta)}{b(t_1,\delta'')} = a(t_0,\eta)$$

$$< a(t,\eta)$$

Therefore $|x(t,t_0,x_k) - p(t)| < \eta$ for any $t > t_1 + \frac{1}{c}\log\frac{b(t_1,\delta'')}{a(t_0,\eta)}$

This contradicts (6).

Thus the proof is completed.

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