The Converse Theorem on Stability

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1 Introduction

The main theorems on stability, asymptotic stability, uniform stability, and uniformly asymptotic stability furnish only sufficient conditions : they contain no hints on how to find a Lyapunov function for a given differential equations. On the other hand, suitable functions can be determined for numerous specific cases and also for certain general types of differential equations. The question then arises as to what extent the main theorems can be reversed, i.e., under what circumstances the existence of a Lyapunov function can be inferred assuming the stability behavior to be known. In other words, under what circumstances are the sufficient conditions expressed by the main theorems also necessary?

The converse theorems are important in studying the properties of solutions of perturbed systems.

The first converse statement of the theorem on stability has been given by K.P. Persidskii [1]. J.L. Massera [2] achieved the first substantial success with respect to asymptotic stability, for which I.G. Malkin [3] [4] previously solved some specific cases. Other converse theorems have been given independently by E.A.Barbasîn, N.N.Krasovskii, J.Kurzweil, V.I.Zubov and T. Yoshizawa.

Recently, we proved the generalization of Lyapunov's stability theorem [5]. The purpose of this paper is to present the converse theorem of our result.

2 Definition and Notations

We shall use the following notations.

Let I denote the interval $0 \leq t < \infty$ and R^n denote Euclidean n-space. For $x \in R^n$, let $|x|$ be the Euclidean norm of x, and D be a domain such that $|x| \leq H$, $H > 0$.

We shall denote by $C_0(x)$ the family of functions which satisfy locally a Lipschitz condition with respect to x.

We consider a system of differential equations

$$\frac{dx}{dt} = f(t,x), \tag{1}$$

where x is an n-dimensional vector and $f(t,x)$ is n-dimensional vector function, which is defined on a region in $I \times R^n$ and is continuous in (t,x) on $I \times D$.

We assume that $f(t,x) \in C_0(x)$ and $f(t,0) \equiv 0$. Throughout this paper a solution through a point (t_0, x_0) in $I \times R^n$ will be denoted by such a form as $x(t, t_0, x_0)$.

We introduce the following definition.

Definition : The zero solution of the system (1) is said to be stable if for any $\varepsilon>0$ and any $t_0\in I$ there exists a $\delta(t_0,\varepsilon)>0$ such that the inequality $|x_0|<\delta$ implies $|x(t,t_0,x_0)|<\varepsilon$ for all $t\geq t_0$.

3 Result

We have the following theorem.

【Theorem】 *In order that the zero solution of the system* (1) *is stable, it is necessary and sufficient that there exists a scalar function* $V(t,x)$ *defined on* $I\times D$ *which satisfies the following conditions* ;

(i) $V(t,0)\equiv 0$ *and* $V(t,x)$ *is continuous in* (t,x) *on* $I\times D$,

(ii) $a(t,|x|)\leq V(t,x)$, *where the function* $a(t,r)$ *is continuous in* (t,r) *on* $I\times I$, $a(t,0)\equiv 0$, $a(t,r)>0$ *for* $r\neq 0$ *and increases monotonically with respect to* t *for each fixed* r,

(iii) $|V(t,x)-V(t,x')|\leq M|x-x'|$, *where* $M>0$ *is a constant,*

(iv) *for any solution* $x(t)$ *of the system* (1), *the function* $V(t,x(t))$ *is a non-increasing function of* t.

Proof. For the proof of the sufficiency, see [5].

We shall see that the condition is necessary. We set

$$V(t,x)=\sup_{\sigma\geq 0}|x(t+\sigma,t,x)|\varphi(t+\sigma)e^{-L\sigma},$$

where $\varphi(t)$ is continuous, monotonically increasing, $\varphi(t)>0$ for $t\neq 0$ and is bounded for bound M, and L is a Lipschitz constant of $f(t,x)$, because the zero solution of the system (1) is stable. Then clearly $f(t,x)$ is continuous in (t,x) on $I\times D$ and we have $|x|\varphi(t)\leq V(t,x)$

Now we shall show that $|V(t,x)-V(t,x')|\leq M|x-x'|$. By Gronwall's Lemma,

$$|x(t+\sigma,t,x)-x(t+\sigma,t,x')|\leq|x-x'|e^{\int_t^{t+\sigma}Lds}=|x-x'|e^{L\sigma}.$$

Therefore, we have

$$\begin{aligned}&|V(t,x)-V(t,x')|\\&=|\sup_{\sigma\geq 0}|x(t+\sigma,t,x)|\varphi(t+\sigma)e^{-L\sigma}-\sup_{\sigma\geq 0}|x(t+\sigma,t,x')|\varphi(t+\sigma)e^{-L\sigma}|\\&\leq M\sup_{\sigma\geq 0}|x(t+\sigma,t,x)-x(t+\sigma,t,x')|e^{-L\sigma}\\&\leq M|x-x'|\end{aligned}$$

Finally, we shall show that $V(t,x(t))$ is a non-increasing function of t. Let $0<t_1<t_2$, $d=t_2-t_1$, we have

$$\begin{aligned}&V(t_2,x(t_2,t_0,x_0))\\&=\sup_{\sigma\geq 0}|x(t_2+\sigma,t_2,x(t_2,t_0,x_0))|\varphi(t_2+\sigma)e^{-L\sigma}\\&=\sup_{\sigma\geq 0}|x(t_2+\sigma,t_0,x_0)|\varphi(t_2+\sigma)e^{-L\sigma}\\&=\sup_{\sigma\geq 0}|x(t_1+d+\sigma,t_0,x_0)|\varphi(t_1+d+\sigma)e^{-L\sigma}\\&\leq\sup_{\sigma\geq 0}|x(t_1+\sigma,t_0,x_0)|\varphi(t_1+\sigma)e^{-L\sigma}\\&=\sup_{\sigma\geq 0}|x(t_1+\sigma,t_1,x(t_1,t_0x_0))|\varphi(t_1+\sigma)e^{-L\sigma}\\&=V(t_1,x(t_1,t_0,x_0)),\end{aligned}$$

hence $V(t,x(t))$ is non-increasing. The theorem is proved.

References

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