Some Result on Asymptotic Stability of Solutions of Ordinary Differential Equations

Shoichi Seino
Miki Kudo
Masamichi Aso
(Received on 31 October, 1978)

1 Introduction

In investigating stable or bounded properties of solutions of a system of ordinary or functional differential equations, the Lyapunov's second method is very useful in that we need not find the solutions of the equations. However, in this method, it is very difficult to find a Lyapunov function which satisfies the conditions for stable or bounded properties in the practical problems. Therefore, it is important to obtain weaker conditions of the Lyapunov function.

In [1] and [2] , we obtained sufficient conditions for stability and asymptotic stability of solutions of a system of differential equations

$$\frac{dx}{dt}=f(t,x) \quad . \tag{1}$$

In a recent paper [3], J.R.Haddock has obtained some sufficient conditions for asymptotic stability of the equation (1) without requiring the boundedness of the function $f(t,x)$.

For asymptotic stability, there are many interesting investigations [4] — [9].

It is the purpose of this paper to give weaker conditions for asymptotic stability.

2 Definitions and Notations

Let R^+ denote the interval $0 \leq t < \infty$ and R^n denote Euclidean n-space.

For $x \in R^n$, let $\| x \|$ be the Euclidean norm of x, and D be a domain such that $\| x \| \leq H$, $H > 0$.

We consider a system of differential equations (1), where x is an n-dimensional vector and $f(t,x)$ is an n-dimensional vector function which is defined on a region in $R^+ \times R^n$, and is continuous in (t,x) on $R^+ \times D$.

Throughout this paper a solution through a point (t_0,x_0) in $R^+ \times R^n$ will be denoted by such a form as $x(t,t_0,x_0)$.

Before stating the definitions we introduce the following notations.

A real-valued function $b(r)$ belongs to class C ($b \in C$) if it is defined and continuous on R^+.

A real-valued function $b(r)$ belongs to class I ($b \in I$) if it is monotone increasing on R^+.

A real-valued function $b(r)$ belongs to class P ($b \in P$) if it satisfies $b(r) > 0$ for $b > 0$ and $b(0) = 0$.

Class with the above three properties is denoted by CIP.

A real-valued function $a(t,r)$ belongs to class $C(t,r)$ ($a \in C(t,r)$) if it is continuous in (t,r).

Class of two variables function which increases monotonically with respect to t for each fixed r is denoted by $I(t \mid r)$ on $R^+ \times R^+$.

We introduce the following definitions.

【Definition 1】

The zero solution of the system (1) is said to be stable if for any $\varepsilon > 0$ and any $t_0 \in R^+$ there exists a $\delta(t_0,\varepsilon) > 0$ such that the inequality $\| x_0 \| < \delta(t_0,\varepsilon)$ implies $\| x(t,t_0,x_0) \| < \varepsilon$ for all $t \geq t_0$.

【Definition 2】

The zero solution of the system (1) is said to be asymptotically stable if it is stable and if there exists a $\delta_0(t_0) > 0$ such that if $\| x_0 \| < \delta_0(t_0)$, $x(t,t_0,x_0)$ tends to zero as $t \to \infty$.

【Definition 3】

Let $V(t,x)$ be a continuous scalar function defined on an open set, and which satisfies locally a Lipschitz condition with respect to x. Corresponding to $V(t,x)$, we define the function

$$V'_{(1)}(t, x) = \lim_{h \to 0+} \frac{1}{h} \{ V(t+h, \ x + hf(t, x)) - V(t, x) \}.$$

In case $V(t, x)$ has continuous partial derivatives of the first order,

it is evident that

$$V'_{(1)}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t, x),$$

where 〝•〟 denotes a scalar product.

3 Preliminary Results

【Theorem 1】

Suppose that there exists a Lyapunov function $V(t,x)$ *defined on* $R^+ \times D$, *which satisfies the following conditions;*

(i) $V(t,0) \equiv 0$,

(ii) $a(t, \|x\|) \leq V(t,x)$, *where* $a \in C(t,r)$, $a \in P$ *with respect to* r *and* $a \in I(t \mid r)$,

(iii) $V'_{(1)}(t,x) \leq 0$.

Then the zero solution of the system (1) *is stable.*

For the proof of this theorem, see [1].

【Theorem 2】

Suppose that the function $f(t,x)$ *is bounded,* $f(t,0) \equiv 0$ *and that there exists a Lyapunov function* $V(t,x)$ *defined on* $R^+ \times D$, *which satisfies the following conditions;*

(i) $V(t,0) \equiv 0$,

(ii) $a(t, \|x\|) \leq V(t,x)$, *where* $a \in C(t,r)$, $a \in P$ *with respect to* r *and* $a \in I(t \mid r)$,

(iii) $V'_{(1)}(t,x) \leq -c(t, \|x\|)$, *where* $c \in C(t,r)$, $c \in P$ *with respect to* r.

Then the zero solution of the system (1) *is asymptotically stable.*

For the proof of this theorem, see [2].

【Theorem 3】

Suppose that there exists function a, b, c∈CIP, *continuous functions* g, h : $R^+ \to R^+ - \{0\}$, *and a Lyapunov function* V(t,x) *such that the following conditions are satisfied;*

(i) $a(\|x\|) \leq V(t,x) \leq g(t)b(\|x\|)$,

(ii) $V'_{(1)}(t,x) \leq -h(t)c(\|x\|)$,

(iii) $\int_0^\infty h(s)c\,[b^{-1}(\frac{\eta}{g(s)})]\,ds = \infty$ *for all small* $\eta > 0$.

Then the zero solution of the system (1) *is asymptotically stable.*

For the proof of this theorem, see [3].

4 Result

【Theorem 4】

Suppose that there exists a Lyapunov function V(t,x) *defined on* $R^+ \times D$, *which satisfies the following conditions;*

(i) $a(t, \|x\|) \leq V(t,x) \leq g(t)b(\|x\|)$, *where* a∈C(t,r),
a∈P *with respect to* r, a∈I(t | r), b∈CIP, $g(t) > 0$ *and* g∈C,

(ii) $V'_{(1)}(t,x) \leq -c(t, \|x\|)$, *where* c∈C(t,r) *and* c∈I(r | t),

(iii) $\int_0^\infty c(s, b^{-1}(\frac{\eta}{g(s)}))\,ds = \infty$ *for all small* $\eta > 0$.

Then the zero solution of the system (1) *is asymptotically stable.*

[Proof]

By Theorem 1, the zero solution of the system (1) is stable.

Let $t_0 \geq 0$ and $\varepsilon > 0$ given $\varepsilon < H$ and let $\delta(t_0, \varepsilon) > 0$ to be chosen such that $\|x_0\| < \delta(t_0, \varepsilon)$ implies $\|x(t,t_0,x_0)\| < \varepsilon$ for all $t \geq t_0$.

Suppose that the zero solution of the system (1) is not asymptotically stable. Then there exists a solution $x(t,t_0,x_0)$ of the system (1) which does not tend to zero as $t \to \infty$. Since $V'(t,x) \leq 0$ along the solution $x(t,t_0,x_0)$, there exists $\eta > 0$ such that $V(t, x(t,t_0,x_0))$ tends to η as $t \to \infty$. This implies $\|x(t,t_0,x_0)\| \geq b^{-1}(\frac{\eta}{g(t)})$ for all $t \geq t_0$. By integrating $V'(t,x)$ along the solution $x(t,t_0,x_0)$, we obtain

$$V(t, x(t,t_0,x_0)) \leq V(t_0,x_0) - \int_{t_0}^t c(s, \|x(s)\|)\,ds$$

$$\leq V(t_0,x_0) - \int_{t_0}^t c(s, b^{-1}(\frac{\eta}{g(s)}))\,ds$$

$$\to -\infty \quad \text{as} \quad t \to \infty.$$

Since $\eta > 0$ can be made sufficiently small by choosing ε small, the above inequality holds and we have a contradiction to $V \geq 0$.

Thus, we see that the zero solution of the system (1) is asymptotically stable.

[Example]

Consider a scalar equation

$$\frac{dx}{dt} = -\frac{x}{t} \qquad (t > 0). \tag{2}$$

For this equation, consider a Lyapunov function $V(t,x)=tx^2$.

Clearly $V'_{(2)}(t,x)=x^2+2tx\frac{dx}{dt}=-x^2$.

We set the functions such that $a(t,r)=\frac{1}{2}tr^2$, $g(t)=2t$, $b(r)=r^2$

and $c(t,r)=\frac{\sqrt{t}}{\sqrt{t}+1}r^2$.

Then the conditions of our theorem is satisfied. Consequently, the zero solution of the system (2) is asymptotically stable.

References

[1] S.Seino, M.Kudo and M.Aso; Some generalization of Lyapunov's stability theorem, Research Reports of Akita Technical College, 8, 1973, 126-131.

[2] S.Seino, M.Kudo and M.Aso; On stability and boundedness of solutions of a system of ordinary differential equations, Research Reports of Akita Technical College, 9, 1974, 93-98.

[3] J.R.Haddock; Some new results on stability and convergence of solutions of ordinary and functional differential equations, Funkcialaj Ekvacioj, 19, 1976, 247-269.

[4] T.Yoshizawa; Stability theory by Lyapunov's second method, Math. Soc. Japan, 1966.

[5] W.Hahn; Theory and application of Liapunov's direct method, Prentice-Hall, Engelewood Cliffs, N.J. 1963.

[6] H.A.Antosiewicz; A survey of Liapunov's second method, Contributions to the Theory of Nonlinear Oscillations, Vol.4,Ann. of Math. Studies, No.41, Princeton Univ. Press, Princeton, N.J.,1958, 141-166.

[7] J.R.Haddock; Some refinements of asymptotic stability theory, Ann. Mat. Pura Appl. 89, 1971, 393-401.

[8] J.R.Haddock; A remark on a stability theorem of M.Marachkoff, Proc. Amer. Math. Soc. 31, 209-212.

[9] R.C.Grimmer and J.R.Haddock; Stability of bounded and unbounded sets for ordinary differential equations, Ann. Mat. Pura Appl. 99, 1974, 143-153.