On the Existence of the Almost Periodic Solutions of the Almost Periodic System

Shoichi Seino
Miki Kudo
Masamichi Aso

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1 Introduction

It is well-known that the boundedness property of solutions of the periodic system implies the existence of a periodic solution. However, for an almost periodic equation, the boundedness of solutions does not necessarily imply the existence of an almost periodic solution even for scalar equations. Opial [8] has constructed an equation with all of its solutions bounded but not almost periodic. Fink and Frederickson [9], by using Opial's equation, have constructed an almost periodic equation which has no almost periodic solutions, but the solutions are uniformly ultimated bounded. Thus, in discussing the existence of an almost periodic solution, several kind of stability properties of a bounded solution were assumed. For example, Miller assumed that the bounded solution is totally stable, and Seifert assumed the Σ-stability of the bounded solution, while Sell assumed the stability under disturbances from the hull. They assumed that the solutions are unique. These results can be obtained by using the property of asymptotically almost periodic functions without the uniqueness of solutions [5], [6].

A basic theorem is as follows due to Coppel.

【Theorem】

Suppose that an almost periodic system

$$x' = f(t, x)$$

has a bounded solution $\pi(t)$ defined on I such that $|\pi(t)| \leq B$ for all $t \geq 0$.

If the solution $\pi(t)$ is asymptotically almost periodic, then the system has an almost periodic solution.

In [1], T. Yoshizawa proved the existence theorem of an almost periodic solution by using Liapunov functions, following an idea of Hale [10].

In this paper, we will state some generalization which weakens the conditions on the Liapunov function.

2 Almost Periodic Functions and Asymptotically Almost Periodic Functions

We shall first list some notations. Let I denote the interval $0 \leq t < \infty$ and R^n denote Euclidean n-space. For $x \in R^n$, let $|x|$ be the Euclidean norm of x. The closure of a set S will be denoted by $\overline{S}$. We shall denote by $C(J \times D, R^n)$ the set of all continuous functions f defined on $J \times D$ with valued in R^n, where J is a subset of R and D is a subset of R^n.

We introduce the following definitions.

〔Definition 1〕

Let $f(t, x) \in C(R \times D, R^n)$, where D is an open set in R^n. $f(t, x)$ is said to be an almost periodic in t uniformly for $x \in D$, if for any $\varepsilon > 0$ and any compact set S in D, there exists a positive number $\ell(\varepsilon, S)$ such that any interval of length $\ell(\varepsilon, S)$ contains a τ for which

$$|f(t+\tau, x) - f(t, x)| \leq \varepsilon$$

for all $t \in R$ and all $x \in S$.

【Lemma 1】

Let $f(t, x) \in C(R \times D, R^n)$ be almost periodic in t uniformly for $x \in D$, where D is an open set in R^n, and let $\{h_k\}$ be a sequence of real numbers. Then, for any $\varepsilon > 0$, and any compact set S in D, there exists a subsequence $\{h_{k_j}\}$ such that the norm of the difference of any pair of functions $f(t+h_{k_j}, x)$, $x \in S$, is less than ε.

Proof. For a given $\varepsilon > 0$, there corresponds an $\ell = \ell(\frac{\varepsilon}{4}, S)$ such that every interval of length ℓ contains an $\frac{\varepsilon}{4}$ -translation number. For each h_k, there exists a τ_k and a γ_k such that $h_k = \tau_k + \gamma_k$, where τ_k is an $\frac{\varepsilon}{4}$ -translation number and $0 \leq \gamma_k \leq \ell$. Since $f(t, x)$ is uniformly continuous on $R \times S$, there is a $\delta(\varepsilon, S) > 0$ such that $|f(t', x) - f(t'', x)| < \frac{\varepsilon}{2}$ if $|t' - t''| < 2\delta$ and $x \in S$. Since $0 \leq \gamma_k \leq \ell$ there exists a subsequence $\{\gamma_{k_j}\}$ of $\{\gamma_k\}$ such that $\gamma_{k_j} \to \gamma$ as $j \to \infty$, where γ is a limit point of the set of all γ_k and consequently $0 \leq \gamma \leq \ell$. Consider the h_{k_j} for which $\gamma - \delta < \gamma_{k_j} < \gamma + \delta$.
Let h_{k_p}, h_{k_m} be two such values. Then we have

$$\sup_{\in R} |f(t+h_{k_p}, x) - f(t+h_{k_m}, x)|$$
$$= \sup |f(t+\tau_{k_p} - \tau_{k_m} + \gamma_{k_p} - \gamma_{k_m}, x) - f(t, x)|$$
$$\leq \sup |f(t+\tau_{k_p} - \tau_{k_m} + \gamma_{k_p} - \gamma_{k_m}, x) - f(t+\gamma_{k_p} - \gamma_{k_m}, x)|$$

$+ \sup|f(t+\gamma_{k_p}-\gamma_{k_m}, x)-f(t, x)|.$

Since $\tau_{k_p}-\tau_{k_m}$ is an $\frac{\varepsilon}{2}$ – translation number and $|\gamma_{k_p}-\gamma_{k_m}|<2\delta$, we have

$$|f(t+h_{k_p}, x)-f(t+h_{k_m}, x)|<\varepsilon$$

for all $t \in R$ and $x \in S$. This proves the lemma.

Let $f(t)$ be a continuous vector function defined on I with values in R^n.

〔Definition 2〕

$f(t)$ is said to be asymptotically almost periodic, if it is a sum of a continuous almost periodic function $p(t)$ and a continuous function $q(t)$ defined on I which tends to zero as $t\to\infty$, that is,

$$f(t)=p(t)+q(t).$$

The concept of asymptotic almost periodicity was introduced by Fréchet [11].

〔Definiton 3〕

We say that $f(t)$ has the property P^*, if given $\varepsilon>0$ there is an $\ell(\varepsilon)$ and a $T(\varepsilon)\geq 0$ such that every interval of length $\ell(\varepsilon)$ contains a τ such that $|f(t+\tau)-f(t)|<\varepsilon$ for $t\geq T(\varepsilon)$ and $t+\tau\geq T(\varepsilon)$.

〔Definition 4〕

We say that $f(t)$ has the property P, if given $\varepsilon>0$ there is an $\ell(\varepsilon)$ and a $T(\varepsilon)\geq 0$ such that every interval of length $\ell(\varepsilon)$ on I contains a τ such that $|f(t+\tau)-f(t)|<\varepsilon$ for $t\geq T(\varepsilon)$.

〔Definition 5〕

We say that $f(t)$ has the property L, if for any sequence $\{h_k\}$ such that $h_k>0$ and $h_k\to\infty$ as $k\to\infty$, when we can select a subsequence $\{h_{k_j}\}$ such that $f(t+h_{k_j})$ converges uniformly on I.

【Lemma 2】

The property P^* is equivalent to the property P.

For proof, see [1].

【Lemma 3】

If $f(t)$, $t\in I$, has the property P, then $f(t)$ has the property L.

Proof. Let $\{h_k\}$ be a sequence such that $h_k>0$ and $h_k\to\infty$ as $k\to\infty$. For a fixed β, $-\infty<\beta\leq 0$, if k is sufficiently large, say $k>K_1$, $f(t+h_k)$ is defined on $\beta\leq t<\infty$ for all k. Since $f(t)$ is bounded and is uniformly continuous for $t\geq 0$, $\{f(t+h_k)\}$ is uniformly bounded and is equicontinuous for $t\geq\beta$. Therefore, there is a subsequence $\{f(t+h'_k)\}$ of $\{f(t+h_k)\}$ which converges to a continuous function $p(t)$ defined on $(-\infty,\infty)$ uniformly on any compact interval in $(-\infty,\infty)$. By the property P, for given $\varepsilon>0$ there exists an $\ell=\ell(\varepsilon)>0$ and a $T(\varepsilon)\geq 0$ and a $\tau_k\in[h'_k-\ell, h'_k]$ such that

$$|f(t+\tau_k)-f(t)|<\varepsilon \quad \text{for} \quad t\geq T(\varepsilon),$$

where τ_k is positive if k is sufficiently large, say $k>K_2$. Let $\ell_k=h'_k-\tau_k$.

Then $0 \leq \ell_k \leq \ell$. Therefore, changing t into $t+\ell_k$, we have

$$|f(t+h'_k) - f(t+\ell_k)| < \varepsilon \quad \text{for} \quad t+\ell_k \geq T(\varepsilon).$$

Since $0 \leq \ell_k \leq \ell$, there exists a subsequence such that

$$\lim_{j\to\infty} \ell_{k_j} = \ell^*, \quad 0 \leq \ell^* \leq \ell.$$

Consider $f(t+h'_k)$ on $[0, \infty)$. If k_j is sufficiently large, then

$$|f(t+h'_{k_j}) - f(t+\ell_{k_j})| < \varepsilon \quad \text{for} \quad t \geq T(\varepsilon),$$

where $t \geq T(\varepsilon)$ and $\ell_{k_j} \geq 0$ imply $t + \ell_{k_j} \geq T(\varepsilon)$.

Since $f(t)$ is uniformly continuous for $t \geq 0$, there is an integer $j_0(\varepsilon) > 0$ such that $j \geq j_0(\varepsilon)$ implies

$$|f(t+\ell_{k_j}) - f(t+\ell^*)| < \varepsilon \quad \text{for} \quad t \geq 0.$$

Thus, if $j \geq j_0(\varepsilon)$ and $t \geq T(\varepsilon)$, we have

$$|f(t+h'_{k_j}) - f(t+\ell^*)| < 2\varepsilon.$$

However, for any t, $f(t+h'_{k_j}) \to p(t)$ as $j \to \infty$, and therefore,

$$|p(t) - f(t+\ell^*)| \leq 2\varepsilon \quad \text{for} \quad t \geq T(\varepsilon).$$

Therefore, $|f(t+h'_{k_j}) - p(t)| < 4\varepsilon$ for $j \geq j_0(\varepsilon)$ and $t \geq T(\varepsilon)$.

On the other hand, for t such that $0 \leq t < T(\varepsilon)$, there is an integer $j'_0(\varepsilon) > 0$ such that if $j \geq j'_0(\varepsilon)$ and $0 \leq t \leq T(\varepsilon)$, then $|f(t+h'_{k_j}) - p(t)| < 4\varepsilon$. Thus, if $j \geq j_0(\varepsilon) + j'_0(\varepsilon)$ and $t \geq 0$,

$$|f(t+h'_{k_j}) - p(t)| < 4\varepsilon.$$

Clearly $j_0(\varepsilon)$ and $j'_0(\varepsilon)$ depend only on ε. This completes the proof.

【Lemma 4】

If $f(t)$, $t \in I$, has the property L, then $f(t)$ is asymptotically almost periodic.

【Lemma 5】

If $f(t)$, $t \in I$, is asymptotically almost periodic, then $f(t)$ has the property P.

For proofs of these lemmas, see references.

Thus we can see that the following three properties are equivalent;

(1) $f(t)$ is asymptotically almost periodic,

(2) $f(t)$ has the property P,

and (3) $f(t)$ has the property L.

【Lemma 6】

If $f(t)$, $t \in I$, has the property P, then the function $p(t)$ in the proof of Lemma 3 is an almost periodic function.

Proof. By Lemma 2, $f(t)$ has the property P*, that is, for any $\varepsilon > 0$ there is an $\ell(\varepsilon) > 0$ and a $T(\varepsilon) \geq 0$ such that every interval of length $\ell(\varepsilon)$ contains a τ such that

$|f(t+\tau)-f(t)|<\varepsilon$ for $t \geq T(\varepsilon)$ and $t+\tau \geq T(\varepsilon)$.

Therefore we have $|f(t+\tau+h_{k_j})-f(t+h_{k_j})|<\varepsilon$ for $t \geq T(\varepsilon)-h_{k_j}$ and $t+\tau \geq T(\varepsilon)-h_{k_j}$. For a fixed $t \in (-\infty,\infty)$, t, $t+\tau \geq T(\varepsilon)-h_{k_j}$ if j is sufficiently large. Letting $j \to \infty$, we have

$|p(t+\tau)-p(t)| \leq \varepsilon$ for all $t \in (-\infty,\infty)$.

This shows that $p(t)$ is almost periodic.

3 Main Result

Throughout this paper a solution through a point (t_0, x_0) in $I \times R^n$ of a system of differential equations

(1) $$x' = f(t, x),$$

where x is an n-dimensional vector and $f(t, x) \in C(I \times D, R^n)$, will be denoted by $x(t, t_0, x_0)$.

〔Definition 6〕

The solution $\pi(t)$ of the system (1) is said to be stable if for any $\varepsilon>0$ and any $t_0 \in I$ there exists a $\delta(t_0, \varepsilon)>0$ such that $|x_0-\pi(t_0)|<\delta(t_0, \varepsilon)$ implies $|x(t, t_0, x_0)-\pi(t)|<\varepsilon$ for all $t \geq t_0$.

〔Definition 7〕

The solution $\pi(t)$ of the system (1) is said to be quasi-equi-asymptotically stable, if for any $\varepsilon>0$ and any $t_0 \in I$, there exists a $\delta_0(t_0)>0$ and a $T(t_0, \varepsilon)>0$ such that if $|x_0-\pi(t_0)|<\delta_0(t_0)$, then $|x(t, t_0, x_0)-\pi(t)|<\varepsilon$ for all $t \geq t_0+T(t_0, \varepsilon)$.

〔Definition 8〕

The solution $\pi(t)$ of the system (1) is said to be equi-asymptotically stable, if it is stable and is quasi-equi-asymptotically stable.

We shall consider a continuous scalar function $V(t, x)$ defined on an open set in $R \times D$. We assume that $V(t, x)$ satisfies locally a Lipschitz condition with respect to x. Corresponding to $V(t, x)$, we define the function

$$V'_{(1)}(t, x) = \overline{\lim_{h \to 0+}} \frac{1}{h}[V(t+h, x+hf(t, x))-V(t, x)].$$

In case $V(t, x)$ has continuous partial derivatives of the first order, it is evident that $V'_{(1)}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t, x)$, where "$\cdot$" denotes the scalar product.

Consider an almost periodic system

(2) $$x' = f(t, x),$$

where $f(t, x) \in C(R \times S_B, R^n)$, $S_B = \{x ; |x|<B\}$, and $f(t, x)$ is almost periodic in t uniformly for $x \in S_B$. In〔1〕, T. Yoshizawa, by using Liapunov functions, proved the existence of an almost periodic solution of the system (2). We will state some generalization of his theorem. In order to discuss

this problem, it is natural to introduce the product system

(3) $$x' = f(t, x), \qquad y' = f(t, y).$$

【Theorem】

Suppose that there exists a Liapunov function $V(t, x, y)$ defined on $0 \le t < \infty$, $|x| < B$, $|y| < B$ which satisfies the following conditions;

(i) $a(t, |x-y|) \le V(t, x, y) \le b(t, |x-y|)$, where the function $a(t, r)$ is continuous in (t, r), $a(t, 0) = 0$, $a(t, r) > 0$ for any $r \neq 0$, and increases monotonically with respect to t and r, and $b(t, r)$ is a continuous function and increases monotonically with respect to r,

(ii) $|V(t, x_1, y_1) - V(t, x_2, y_2)| \le K(|x_1 - x_2| + |y_1 - y_2|)$, where $K > 0$ is a constant,

(iii) $V'_{(3)}(t, x, y) \le -\alpha V(t, x, y)$, where $\alpha > 0$ is a constant.

Moreover, suppose that there exists a solution $\pi(t)$ of (2) such that $|\pi(t)| \le B^* < B$ for $t \ge 0$. Then, in the region $R \times S_B$, there exists a unique equi-asymptotically stable almost periodic solution $p(t)$ of (2) which is bounded by B^*. In particular, if $f(t, x)$ is periodic in t of period ω, then there exists a unique equi-asymptotically stable periodic solution of (2) of period ω.

Proof. Let $\{\tau_k\}$ be a sequence such that $\tau_k \to \infty$ as $k \to \infty$. Set $\pi_k(t) = \pi(t + \tau_k)$. Then $\pi_k(t)$ is a solution of $x' = f(t + \tau_k, x)$ through $(0, \pi(\tau_k))$. Since $f(t, x)$ is almost periodic, there exists a subsequence of $\{\tau_k\}$, which we shall denote by $\{\tau_k\}$ again, such that $f(t + \tau_k, x)$ converges uniformly on $R \times \bar{S}_{B^*}$ as $k \to \infty$. For a given $\varepsilon > 0$, choose an integer $k_0(\varepsilon)$ so large that if $m \ge k \ge k_0(\varepsilon)$, by (i) and Lemma 1,

(4) $$b(t_0, 2B^*)\, e^{\alpha(t_0 - \tau_k)} < \frac{a(t_0, \varepsilon)}{2}$$

and

(5) $$|f(t+\tau_k, x) - f(t+\tau_m, x)| < \frac{a(t, \varepsilon)\alpha}{2K} \qquad \text{on } R \times \bar{S}_{B^*}.$$

From conditions (ii) and (iii), it follows that

$$V'(t, \pi(t), \pi(t+\tau_m-\tau_k)) \le -\alpha V(t, \pi(t), \pi(t+\tau_m-\tau_k)) + K|f(t+\tau_m-\tau_k, \pi(t+\tau_m-\tau_k)) - f(t, \pi(t+\tau_m-\tau_k))|.$$

By (5), we have

$$V'(t, \pi(t), \pi(t+\tau_m-\tau_k)) \le -\alpha V(t, \pi(t), \pi(t+\tau_m-\tau_k)) + \frac{a(t_0, \varepsilon)\alpha}{2},$$

which implies that

$$V(t+\tau_k, \pi(t+\tau_k), \pi(t+\tau_m)) \le e^{\alpha(t_0 - t - \tau_k)}\, V(t_0, \pi(t_0), \pi(t_0+\tau_m-\tau_k)) + \frac{a(t_0, \varepsilon)}{2}.$$

Thus, if $m \ge k \ge k_0(\varepsilon)$, by (4),

$$V(t+\tau_k, \pi(t+\tau_k), \pi(t+\tau_m)) < a(t_0, \varepsilon).$$

Therefore, by (i), we have $|\pi(t+\tau_k) - \pi(t+\tau_m)| < \varepsilon$ for all $t \ge 0$ if $m \ge k \ge k_0(\varepsilon)$, which shows that $\pi(t)$ has the property L. Since the property L is equivalent to the property P, by Lemma 6, the system (2) has an almost periodic solution $p(t)$ which is bounded by B^*. By [12], solution $p(t)$ of (2) is stable, and hence, there exists a $\delta_0(t_0, B)>0$ such that if $t_0\in I$ and $|x_0-p(t_0)|<\delta_0(t_0, B)$, then

$$|x(t, t_0, x_0)-p(t)|<B \quad \text{for all } t\geq t_0.$$

Moreover, for any $\varepsilon>0$, there exists a $\delta(t_0, \varepsilon)>0$ such that if $t_0\in I$ and $|x_0-p(t_0)|<\delta(t_0, \varepsilon)$,

$$|x(t, t_0, x_0)-p(t)|<\varepsilon \quad \text{for all } t\geq t_0.$$

Now we show that every solution $x(t, t_0, x_0)$ of (2) such that $t_0\in I$, $|x_0-p(t_0)|<\delta_0(t_0, B)$, satisfies $|x(t, t_0, x_0)-p(t)|<\delta(t_0, \varepsilon)$ at some time t.

Put $t=t_0+T(t_0, \varepsilon)$, where $T(t_0, \varepsilon)=\dfrac{1}{\alpha}\log\dfrac{b(t_0, \delta_0(t_0, B))}{a(t_0, \delta(t_0, \varepsilon))}$.

By (i),

$$a(t_0, |x(t, t_0, x_0)-p(t)|)\leq V(t, x(t, t_0, x_0), p(t))$$
$$\leq b(t_0, |x_0-p(t_0)|)e^{-\alpha(t-t_0)}< b(t_0, \delta_0(t_0, B))\cdot\frac{a(t_0, \delta(t_0, \varepsilon))}{b(t_0, \delta_0(t_0, B))}$$
$$=a(t_0, \delta(t_0, \varepsilon)),$$

and hence

$$|x(t, t_0, x_0)-p(t)|<\varepsilon \quad \text{for all } t\geq t_0+T.$$

This shows that the solution $p(t)$ is quasi-equi-asymptotically stable. Thus the solution $p(t)$ is equi-asymptotically stable. Every solution remaining in S_B approaches $p(t)$ as $t\to\infty$, which implies the uniqueness of $p(t)$. In a case which $f(t, x)$ is periodic in t of period ω, $p(t+\omega)$ is also a solution of (2) which remains in S_B, and hence $p(t+\omega)\to p(t)$ as $t\to\infty$. Thus we have $p(t+\omega)=p(t)$. This completes the proof.

Remark. If $a(t, r)$ and $b(t, r)$ in Theorem are independent of t, our result implies Yoshizawa's theorem.

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