ON 4-DIMENSIONAL SUBMANIFOLDS OF A SPACE OF CONSTANT CURVATURE

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1. Introduction

Let M^4 be a 4-dimensional Riemannian manifold. If M^4 is an Einstein space, we have

(*) $K(e_1, e_2) = K(e_3, e_4)$, $K(e_1, e_3) = K(e_2, e_4)$, $K(e_1, e_4) = K(e_2, e_3)$

for any orthonormal vector fields e_i in M^4, where $K(e_i, e_j)$ is sectional curvature corresponding to the plane spanned by $\{e_i, e_j\}$.

The purpose of this paper is to consider the submanifold M^4 of a space of constant curvature satisfying the condition (*).

2. Submanifolds

Let $R^{n+4}(k)$ be an $n+4$-dimensional Riemannian manifold of constant curvatnre k. Let M^4 be a submanifold of codimension n immersed in $R^{n+4}(k)$. Let g (resp. $\tilde{g}$) be a Riemannian metric and ∇ (resp. $\tilde{\nabla}$) be a Riemannian connectin on M^4 (resp. $R^{n+4}(k)$).

Let $\xi_1, \cdots\cdots, \xi_n$ be orthonormal normal vector fields of M^4 and h^i be the corresponding second fundamental forms. Then we may write, for any vector fields X, Y and any normal vector field ξ on M^4,

(1) $\tilde{\nabla}_X Y = \nabla_X Y + h^i(X, Y)\xi_i, \qquad \tilde{\nabla}_X \xi = -A_\xi(X) + \nabla_X^\perp \xi$

where $\nabla_X Y$ (resp. $-A_\xi(X)$) is a vector field tangent to M^4 and $h^i(X, Y)\xi_1$ (resp. $\nabla_X^\perp \xi$) is a vector field normal to M^4. If we put $A^i = A_{\xi_i}$, then we have

(2) $h^i(X, Y) = g(A^i(X), Y)$.

Let R be a curvature tensor on M^4, then

(3) $R(X, Y; Z, W) = k\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} + g(h^i(X, W)\xi_i, h^i(Y, Z)\xi_i) - g(h^i(X, Z)\xi_i, h^i(Y, W)\xi_i)$

where $R(X, Y; Z, W) = g(R(X, Y)Z, W)$.

If we denote dy R^N the curvature tensor of the normal connection $\nabla^\perp$ on the normal bundle $T^\perp(M^4)$, that is,

(4) $R^N(X, Y)\xi = \nabla_X^\perp \nabla_Y^\perp \xi - \nabla_Y^\perp \nabla_X^\perp \xi - \nabla_{[X, Y]}^\perp \xi$,

then we have

(5) $R^N(X, Y; \xi, \eta) = g([A_\xi, A_\eta](X), Y)$

for any vector fields X, Y and any normal vector fields ξ, η on M^4.

Let X and Y be two linearly independent vector at a point $p \in M^4$ and

$\gamma(X, Y)$ be the plane section by X and Y. The sectional curvature $K(X, Y) \equiv K(\gamma)$ for γ is defined by

$$(6) \qquad K(X, Y) = -\frac{R(X, Y; X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}$$

It is easy to see that this $K(\gamma)$ is uniquely determined by the plane section γ and is independent of the choice of X and Y on it. If $K(\gamma)$ is a constant for all sections γ in the tangent space $T_p(M^4)$ at p and for all points $p \in M^4$, then M^4 is called a space of contant curvature.

We consider endoomorphism $R(X, Y)$ operates on R as a derivation of tensor algebra at each point of M^4, then we have

$$(7) \qquad (R(X, Y) \cdot R)(Z, W) = [R(X, Y), R(Z, W)] - R(R(X, Y)Z, W) - R(Z, R(X, Y)W)$$

for any vector fields X, Y, Z, W on M^4. And

$$(8) \qquad R(X, Y) = kX\wedge Y + \sum_{i=1}^{n} A^i(X)\wedge A^i(Y)$$

where $(X\wedge Y)Z = g(Y, Z)X - g(X, Z)Y$.

3. Flat normal connection

If the curvature tensor R^N of the normal connection $\nabla^\perp$ vanishes, that is, $R^N(X, Y) = 0$ for any vector fields X, Y on M^4, then the normal connection $\nabla^\perp$ is said to be flat. It is wellknown that the normal connection $\nabla^\perp$ of M^4 in $R^{n+4}(k)$ is flat if and only if there exist locally n mutually orthogonal unit normal vector fields ξ_q such that each of the ξ_q is parallel in the normal bundle. Furthermore the normal connection $\nabla^\perp$ is flat if and only if all the second fundamental tensors A^q are simultaneously diagonalizable. Hence, we may choose orthonormal vector fields e_i on M^4 such that the e_i are the principal directions with respect to each of the ξ_q with principal curvatures $a^q{}_i$, respectively. Then, by (2), (3) and (6),

$$(9) \qquad K(e_i, e_j) = k + \sum_{q=1}^{n} a_i{}^q a_j{}^q$$

We put $a = \sum_{q=1}^{n} a_1{}^q a_2{}^q$, $\quad b = \sum_{q=1}^{n} a_1{}^q a_3{}^q$, $\quad c = \sum_{q=1}^{n} a_1{}^q a_4{}^q$.

[Theorem 1]

Let M^4 be a submanifold of codimension n immersed in $R^{n+4}(k)$ and satisfying the condition (*) and $a \neq -k$, $b \neq -k$, $c \neq -k$. Suppose that the normal connection is flat. If curvature tensor R satisfies $R(X, Y) \cdot R = 0$ for all tangent vectors X and Y, then M^4 is a space of constant curvature.

Proof. From (*), $R(X, Y) \cdot R = 0$, (7), (8), we have

$$(10) \qquad (a+k)(b-c) = 0, \quad (b+k)(c-a) = 0, \quad (c+k)(a-b) = 0.$$

Let Ric be a ricci curvature tensor on M^4, then we have

$$(11) \qquad Ric(e_i, e_i) = \sum_{j=1}^{4} R(e_j, e_i; e_i, e_j) = 3k + a + b + c$$

$$Ric(e_i, e_j) = 0 \qquad \text{for } i \neq j.$$

Therefore $Ric(X, Y) = (3k + a + b + c)g(X, Y)$, that is, M^4 is an Einstein space. From above, $3k+a+b+c$ is constant. From (10), $\lambda \equiv a=b=c$ (constant). Therfore we can see that

$K(e_1, e_2) = K(e_1, e_3) = K(e_1, e_4) = k+\lambda$ is constant.

4. Umbilical sudmanifolds

We have the following lemma

[Lemma] [3]

Any n-dimensional pseudo-umbilical submanifold M^n of codimension 2 immersed in a space M^{n+2} of constant curvature is conformally flat, if $n \geqq 4$.

Let M^4 be a submanifold of codimension 2 immersed in a space $R^6(k)$. Let ξ, η be orthonormal normal normal vector fields of M^4 and A^1 (resp. A^2) be the second fundamental tensor with rfspect to ξ (resp. η).

[Theorem 2]

Let M^4 be a submanifold of codimension 2 immersed in $R^6(k)$ and satisfying the condition (*). If M^4 is umbilical with respect to ξ, then M^4 is a space of constant curvature.

Proof. We choose orthonormal vector fields e_i such that A^2 is diagonalizable and let a_i be principal curvature wish respect to η.

From assumpsion, there exists a function μ such that $A^1(X) = \mu X$ for any vector field X on M^4. We put $a=\mu^2+a_1a_2$, $b=\mu^2+a_1a_3$, $c=\mu^2+a_1a_4$.

From (*), we have

(12) $a_1a_2 = a_3a_4, \quad a_1a_3 = a_2a_4, \quad a_1a_4 = a_2a_3.$

From (12), we have only to consider the following cases.

Ⅰ. $a_1 = a_2 = a_3 = a_4 \neq 0$

Ⅱ. i) $a_1 = a_2 = -a_3 = -a_4$, ii) $a_1 = -a_2 = -a_3 = a_4$, iii) $a_1 = -a_2 = a_3 = -a_4$

Ⅲ. i) $a_2 = a_3 = a_4 = 0$, $a_1 \neq 0$ ii) $a_1 = a_3 = a_4 = 0$, $a_2 \neq 0$

iii) $a_1 = a_2 = a_4 = 0$, $a_3 \neq 0$ Ⅳ) $a_1 = a_2 = a_3 = 0$, $a_4 \neq 0$

Ⅳ. $a_1 = a_2 = a_3 = a_4 = 0$.

Case Ⅰ., then M^4 is totally umbilical. Therfore M^4 is a space of constant curvature.

Case Ⅱ., then M^4 is pseudo-umbilical. From Lemma, M^4 is conformally flat. On the other hand, from $Ric(e_i, e_i) = 3k+a+b+c$ and $Ric(e_i, e_j) = 0$ for $i \neq j$, M^4 is an Einstein space. Therfore M^4 is a space of constant curvasure.

Cases Ⅲ Ⅳ., then, since M^4 is an Einstein space, $K(e_i, e_j) = k+\mu^2$ is constant. Therfore M^4 is a space of constant curvasure.

References

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