Finite Element Approximation for Eigenvalues of Vibrating Plate

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1. Approximation procedure

We concider the probrem of determining the free vibration of a thin plate. The motion of the plate is governed by the following partial diffential equation of fourth oder

$$w_{xxxx}+2w_{xxyy}+w_{yyyy}+w_{tt}=0,$$

where w (x, y, t) is the transverse displacement of the plate.

We seek solutions of the form $w(x, y, t)=u(x, y)f(t)$. Setting this in the equation gives $\Delta\Delta uf(t)+u(x, y)f_{tt}=0$, or $\Delta\Delta u/u=-f_{tt}/f=\lambda$, so that $\Delta\Delta u-\lambda u=0$, or $Lu=\lambda u$, where L denotes the iterated Laplacian $\Delta\Delta$.

In order to determine suitable boundary condions, let R denote a bounded region of the plane with boundary C, and multiply Lu by a function v and integrate by parts over R

$$(Lu, v)=\iint_R v\Delta\Delta u dxdy=\iint_R \Delta v\Delta u dxdy+\int_C v\frac{\partial\Delta u}{\partial n}ds-\int_C\frac{\partial v}{\partial n}\Delta u ds,$$

where $\frac{\partial}{\partial n}$ denote the outward normal derivative. Thus the boundary conditions, prescribed and natural, must be such that

$$\int_C\left(v\frac{\partial\Delta u}{\partial n}-\frac{\partial v}{\partial n}\Delta u\right)ds=0.$$

The system formed by $L=\Delta\Delta$ with these conditions is self-adjoint, since

$(Lu, v)=\iint_R\Delta u\Delta v dxdy=(u, Lv)$, and the operator L is formaly posive definite.

If we prescribe $u=0$ (supported plate), the corresponding natural condition is $\Delta u=0$. If we prescribe $u=0$ and $\frac{\partial u}{\partial n}=0$ everywhere on C, the plate is clamped, and there are no natural conditions.

Now we have obtained the Galerkin weak form of the eigenvalue problem $Lu=\lambda u$: Find a scalar λ and a function u in the admissible space H^2_E such that

$$a(u, v)=\lambda(u, v)$$

holds for all v in H^2_E, where $a(u, v)=(Lu, v)=\iint_R\Delta u\Delta v dxdy=(\Delta u, \Delta v)$.

Since the operator L is symmetric, the eigenvalues are real.

To approximate eigenvalues we work only within a finite-dimensional subspace (finite element space) S^h of the full admissible space H^2_E • In this space we look for a pair λ^h and u^h such that

(*) $a(u^h, v^h) = \lambda^h (u^h, v^h)$ for all v^h in S^h.

Choose a basis $\varphi_1 \cdots\cdots, \varphi_N$ for S^h, then any v^h in S^h can be expanded as

$$v^h = \sum_j q_j \varphi_j$$

where q_j are the generalized co-ordinates (the nodal parameters of v^h).

Substituting $u^h = \sum_j q_j \varphi_j$ into (*) and taking $v^h = \varphi_k$ we have

$$a(\sum q_j \varphi_j, \varphi_k) = \lambda^h (\sum q_j \varphi_j, \varphi_k),$$

that is

$$\sum q_j a(\varphi_j, \varphi_k) = \lambda^h \sum q_j (\varphi_j, \varphi_k).$$

This is simply the k-th row of the matrix equation

$$KQ = \lambda^h MQ.$$

We note that the mass matrix M is symmetric and positive definite; it is the Gram matrix for the linear indendent vectors $\varphi_1, \cdots\cdots\cdots, \varphi_N$.

The eigenvalues $\lambda^h{}_k$ are expected to approximate the continuous eigenvalues λ_k and the eigenvectors Q_k lead to a corresponding approximate eigenfunction

$$u^h_k = \sum_{j=1}^{N} (Q_k)_j \varphi_j.$$

Thus the components of the discrete eigenvectors in the matrix problem $KQ = \lambda MQ$, yield the nodal values of the finite element eigenfunctions.

It is known that the eigenfunctions are the points u_k at which the Rayleigh quotient $R(v) = \dfrac{a(v, v)}{(v. v)}$ is stationary, and the corresponding eigenvalues are $\lambda_k = R(u_k)$. So that $\lambda^h{}_1$ always lies above λ_1 ; $\lambda_1{}^h \geqq \lambda_1$, since $\lambda_1{}^h$ is the minimum values of $R(v)$ over the subspace S^h and λ_1 is the minimum over the whole admissible space H^2_E.

2. Finite element

In order to construct a reduced quintic triangular element, consider a triangle with nodes 1, 2, 3, whose Cartesian coordinates are (x_i, y_i).

We construct a basis such that there are eighteen nodal parameters given by v, v_x, v_y, v_{xx}, v_{xy}, and v_{yy} at the vertices. To do so we introduce area coordintes as follows:

Let $\alpha_i = x_j y_k - x_k y_j$, $\beta_i = y_j - y_k$, $\gamma_i = x_k - x_j$, where i, j, k are a cyclic permutation of 1, 2, 3, and let

$$a_i = \alpha_i/\alpha, \quad b_i = \beta_i/\alpha, \quad c_i = \gamma_i/\alpha \qquad (i = 1,\ 2,\ 3)$$

where $\alpha = \alpha_1 + \alpha_2 + \alpha_3$. With these a_i, b_i, c_i, area coordintes ω_i corresponding to Cartesian coordinate (x, y) are defined as

$$\omega_i = a_i + b_i x + c_i y.$$

Then the basis functions $\{\varphi_m\}$ are introduced as follows:

$$\{\varphi_m\} = [p_{mn}]\{f_n\} \qquad (m = 1, 2, \cdots\cdots, 18;\ n = 1, 2, \cdots\cdots, 21)$$

here

$$[p_{mn}]=\begin{pmatrix} D_1 & & & M_1 \\ & D_2 & & M_2 \\ & & D_3 & M_3 \end{pmatrix}$$

where

$$D_i=\begin{pmatrix} 1 & 5 & 5 & 10 & 20 & 10 \\ & \gamma_k & -\gamma_j & 4\gamma_k & 4(\gamma_k-\gamma_j) & -4\gamma_j \\ & -\beta_k & \beta_j & -4\beta_k & -4(\beta_k-\beta_j) & 4\beta_j \\ & & & 0.5\gamma^2_k & -\gamma_j\gamma_k & 0.5\gamma^2_j \\ & & & -\beta_k\gamma_k & \beta_j\gamma_k+\beta_k\gamma_j & -\beta_k\gamma_j \\ & & & 0.5\beta^2_k & -\beta_j\beta_k & 0.5\beta^2_j \end{pmatrix}$$

$$M_1=\begin{pmatrix} & 15 & 15 \\ & 3\gamma_3-4.5\gamma_2 & 4.5\gamma_3-3\gamma_2 \\ & 4.5\beta_2-3\beta_3 & 3\beta_2-4.5\beta_3 \\ & \gamma^2_2/4-\gamma_2\gamma_3 & \gamma^2_3/4-\gamma_2\gamma_3 \\ & \beta_2\gamma_3+\beta_3\gamma_2-0.5\beta_2\gamma_2 & \beta_2\gamma_3+\beta_3\gamma_2-0.5\beta_3\gamma_3 \\ & \beta_2^2/4-\beta_2\beta_3 & \beta_3^2/4-\beta_2\beta_3 \end{pmatrix}$$

$$M_2=\begin{pmatrix} 15 & & 15 \\ 4.5\gamma_1-3\gamma_3 & & 3\gamma_1-4.5\gamma_3 \\ 3\beta_3-4.5\beta_1 & & 4.5\beta_3-3\beta_1 \\ \gamma_1^2/4-\gamma_1\gamma_3 & & \gamma_3^2/4-\gamma_1\gamma_3 \\ \beta_1\gamma_3+\beta_3\gamma_1-0.5\beta_1\gamma_1 & & \beta_1\gamma_3+\beta_3\gamma_1-0.5\beta_3\gamma_3 \\ \beta_1^2/4-\beta_1\beta_3 & & \beta_3^2/4-\beta_1\beta_3 \end{pmatrix}$$

$$M_3=\begin{pmatrix} 15 & 15 & \\ 3\gamma_2-4.5\gamma_1 & 4.5\gamma_2-3\gamma_1 & \\ 4.5\beta_1-3\beta_2 & 3\beta_1-4.5\beta_2 & \\ \gamma_1^2/4-\gamma_1\gamma_2 & \gamma_2^2/4-\gamma_1\gamma_2 & \\ \beta_1\gamma_2+\beta_2\gamma_1-0.5\beta_1\gamma_1 & \beta_1\gamma_2+\beta_2\gamma_1-0.5\beta_2\gamma_2 & \\ \beta_1^2/4-\beta_1\beta_2 & \beta_2^2/4-\beta_1\beta_2 & \end{pmatrix}$$

and

$$\{f_n\}=\Big(\omega_1^5 \;\; \omega_1^4\omega_2 \;\; \omega_1^4\omega_3 \;\; \omega_1^3\omega_2^2 \;\; \omega_1^3\omega_2\,\omega_3 \;\; \omega_1^3\omega_3^2 \;\; \omega_2^5 \;\; \omega_2^4\omega_3 \;\; \omega_2^4\omega_1 \;\; \omega_2^3\omega_3^2 \;\; \omega_2^3\omega_3\,\omega_1 \;\; \omega_2^3\omega_1^2$$
$$\omega_3^5 \;\; \omega_3^4\omega_1 \;\; \omega_3^4\omega_2 \;\; \omega_3^3\omega_1^2 \;\; \omega_3^3\omega_1\,\omega_2 \;\; \omega_3^3\omega_2^2 \;\; \omega_1\,\omega_2^2\omega_3^2 \;\; \omega_1^2\omega_2\,\omega_3^2 \;\; \omega_1^2\omega_2^2\omega_3\Big)^t.$$

3. Numerical results

Consider the problem of a vibration of a square plate with sides $x=\pm 1$, $y=\pm 1$. For the supported plate with sides a and b, it is known that the eigenvalues are $\lambda_{mn}=(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2)^2$ and the corresponding eigenfunctions are $u_{mn}(x, y)=\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$. To compute the matrix eigenvalues we employed the generalized Jacobi method. The numerical results for the first three eigenvalues are as follows:

subdivision \ boundary condition	supported	clumped
Reduced cubic triangular element		
2×2	18.03 186.1 186.1	62.74 379.1 379.1
4×4	21.60 138.1 138.1	78.00 346.3 346.3
Reduced quintic triangular element		
2×2	29.09 175.7 175.7	69.37 277.5 277.5
Hermite bicubic rectangular element		
2×2	24.46 175.5 175.5	85.90 563.1 563.1
3×3	24.37 154.7 154.7	81.92 353.5 353.5
4×4		81.26 342.9 342.9
exact	24.35 152.2 152.2	

(For the basis of the reduced cubic triangular element and the rectangular element, see [2].)

These numerical experiments have been performed on the HITAC 8250-48 at Akita Technical College Computer Center.

References

1) G.Strang and G.J.Fix : An Analysis of the Finite Element Method. PrenticeHall (1973).

2) Takashi Yoshimura : Finite Element Solution of Bending of Plate. Research Reports of Akita Technical College No.10 (1975).