ON A COMPLEX HYPERSURFACE OF A K-SPACE

Fumio NARITA

(Received on 31 October, 1975)

1. Introduction

B. Smyth proved in his thesis [1] the following

[Theorem] Let M be a complex hypersurface of a Kählerian manifold $\tilde{M}$ of constant holomorphic sectional curvature. If M is Einstein manifold it is locally symmetric.

In this paper, we shall prove that an Einstein complex hypersurface of a irreducible symmetric K-space is locally symmetric.

2. *O-spaces and K-spaces

Let $(\tilde{M},J,g)$ be an almost Hermitian manifold of complex dimension $n+1$, and denote the almost complex structure and the Hermitian metric of $\tilde{M}$ by J and g respectively. By $\tilde{\nabla}$ we alwayo mean the Riemannian covariant differentiation on $\tilde{M}$. An almost Hermitian manifold $\tilde{M}$ is called an *O-space (or quasi-Kählerian manifold) or K-spce (or Tachibana space or nearly Kähler manifold) according as

(2.1) $(\tilde{\nabla}_X J)Y+(\tilde{\nabla}_{JX}J)JY=0$

or

(2.2) $(\tilde{\nabla}_X J)Y+(\tilde{\nabla}_Y J)X=0$ (or epuivalently $(\tilde{\nabla}_X J)X=0$)

holds for any vector fields X and Y on $\tilde{M}$. It is well known that a K-space is an *O-space.

3. Complex hypersurface of a K-space

Let $(\tilde{M},J,g)$ be an almost Hermitian manifold of complex dimension $n+1$.

Let M be a complex hypersurface of $\tilde{M}$, i.e., suppose that there exist a complex analytic mapping $f:M\rightarrow\tilde{M}$. Then for each $x\in M$ we identity the tangent space $T_x(M)$ with $f_*(T_x(M))(T_{f(x)}(\tilde{M})$ by means of f_*. Sinde $f^*\circ g=g'$ and $J\circ f_*=f_*\circ J'$ where g' and J' are the Hermitian metric and the almost complex structure of M respectively, g' and J' are respectively identified with the restrictions of the structures g and J to the subspace $f_*(T_x(M))$. As is well known, we can choose the following special neighborhood U(x) of x fof a neighborhood $\tilde{U}(f(x))$ of f(x). Let $\{\tilde{U};\tilde{x}^i\}$ $(i=1, \cdots, 2n+2)$ be a system of coordinate neighborhoods of $\tilde{M}$. Then

$\{U;x^i\}$ is a system of coordinate neighborhoods of M such that $x^{2n+1}=x^{2n+2}=0$ where $x^i=\tilde{x}^i \circ f$.

We have following lemmas

[Lemma 3.1[2]] A complex hypersurface M of a K-space $\tilde{M}$ is also a K-space.

Let ξ be a differentiable unit vector field normal to M at each point of U(x). If X and Y are vector fields on the neighborhood U(x), we may write

(3.1) $\tilde{\nabla}_X Y=\nabla_X Y+h(X,Y)\xi+k(X,Y)J\xi$,

where $\nabla_X Y$ denotes the component of $\tilde{\nabla}_X Y$ tangent to M.

The identity $g(\xi,\xi)=1$ implies $g(\tilde{\nabla}_X\xi,\xi)=0$ on U(x) for any vector field X on U(x). We may therefore write

(3.2) $\tilde{\nabla}_X\xi=-A(X)+(X)J\xi$,

where A(X) is tangent to M.

[Lemma 3.2[2]] In a compelx hypersurface M of a K-space $\tilde{M}$, at any point $y\in U(x)$ there exists an orthonormal basis $\{e_1,\cdots, e_n, Je_1,\cdots, Je_n\}$ of $T_y(M)$ with respect which the matrix A is diagonal of the form

$$\begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & -\lambda_1 & & \\ & & & & \ddots & \\ & & & & & -\lambda_n \end{pmatrix}$$

where $Ae_i=\lambda_i e_i$, and $AJe_i=-\lambda_i Je_i$, $i=1,\cdots, n$.

[Lemma 3.3[2]] If $\tilde{R}$ and R are the Riemannian curvature tensors of a K-space $\tilde{M}$ and a complex hypersurface M of $\tilde{M}$ respectively, then for any vector fields X,Y,Z and W on U(x) we have the following Gauss equation.

$$\tilde{R}(X,Y,Z,W)=R(X,Y,Z,W)$$
$$(3.3) \quad -\{g(AX,Z)g(AY,W)-g(AX,W)g(AY,Z)\}$$
$$-\{g(JAX,Z)g(JAY,W)-g(JAX,W)g(JAY,Z)\}$$

[Lemma 3.4] For any pair of vectors X and Y tangent to M at a point of U(x), we have the equations

(3.4) $(\nabla_X A)Y-(\nabla_Y A)X-s(X)JAY+s(Y)JAX=0$

(3.5) $(\nabla_X JA)Y-(\nabla_Y JA)X+s(X)AY-s(Y)AX=0$

(3.6) $Ric(X,Y)=-2g(A^2X,Y)+\tilde{R}ic(X,Y)$

[Lemm 3.5[1]] If M is an arbitrary Riemannian manifold with metric g, then the tensor feild P on M defined by

$$P(X,Y,Z,W)=g(BX,Z)g(BY,W),$$

where B is a tensor field of type (1,1) on M, has covariant differential given by

$$(\nabla_V P)(X,Y,Z,W)=g((\nabla_V B)X,Z)g(BY,W)+g(BX,Z)g((\nabla_V B)Y,W).$$

[Theorem 1] Let M be a complex hypersurface of an Einstein K-space. If M is an Einstein manifold, then

$$\nabla_X A=s(X)JA \text{ and } \nabla_X(JA)=-s(X)A$$

where X is any vector tangent to M at any point of U(x).

Proof. Consider the distributions T_+ and T_- on $U(x)$ defined by

$$T_+(y)=\{X\in T_y(M)\mid AX=\lambda X\}$$
$$T_-(y)=\{X\in T_y(M)\mid AX=-\lambda X\}\qquad \text{for each } y\in U(x).$$

Since $A^2=\lambda^2 I$ and λ is a constant (see (3.6)),

$$0=(\nabla_X(AA))Y=A(\nabla_X A)Y+(\nabla_X A)AY$$

for $X,Y\in T_y(M)$ and $y\in U(x)$. If $Y\in T_+(y)$ then

$$A(\nabla_X A)Y+\lambda(\nabla_X A)Y=0$$

Hence $(\nabla_X A)Y\in T_-(y)$ if $Y\in T_+(y)$, and similarly $(\nabla_X A)Y\in T_+(y)$ if $Y\in T_-(y)$

By (3.4)

$$(\nabla_X A)Y=s(X)JAY \qquad \text{where } X\in T_-(y),\ Y\in T_+(y)$$

and

$$(\nabla_X A)Y=s(X)JAY \qquad \text{where } X\in T_+(y),\ Y\in T_-(y).$$

If $X\in T_-(y)$ we have

$$(\nabla_X A)X=-(\nabla_X A)JJX=J(\nabla_X A)JX+(\nabla_X J)AJX+A(\nabla_X J)JX$$
$$=J(\nabla_X A)JX+\lambda(\nabla_X J)JX+A(\nabla_X J)JX$$

From (2.1) and (2.2) we have $(\nabla_X J)JX=0$

Hence $(\nabla_X A)X=J(\nabla_X A)JX=s(X)JAX$.

Thus if $X\in T_-(y)$(resp. $T_+(y)$) and $Y\in T_-(y)$(resp. $T_+(y)$) we find

$$(\nabla_{X+Y}A)(X+Y)=s(X+Y)JA(X+Y)$$
$$=s(X)JAX+s(X)JAY+s(Y)JAX+s(Y)JAY$$
$$(\nabla_{X+Y}A)(X+Y)=(\nabla_X A)X+(\nabla_X A)Y+(\nabla_Y A)X+(\nabla_Y A)Y$$
$$=s(X)JAX+(\nabla_X A)Y+(\nabla_Y A)X+s(Y)JAY$$

Therefore

$$(\nabla_X A)Y+(\nabla_Y A)X=s(X)JAY+s(Y)JAX.$$

On the other hand, by (3.4)

$$(\nabla_X A)Y-(\nabla_Y A)X=s(X)JAY-s(Y)JAX.$$

Hence $(\nabla_X A)Y=s(X)JAY$.

We prove the other formula.

If $X\in T_-(y)$ we have

$$(\nabla_X JA)X=-(\nabla_X A)JX-A(\nabla_X JX)+AJ(\nabla_X X)$$
$$=-s(X)JAJX-A((\nabla_X J)X)=-s(X)AX$$

If $X\in T_-(y)$(resp. $T_+(y)$) and $Y\in T_-(y)$(resp. $T_+(y)$) we find

$$(\nabla_{X+Y}JA)(X+Y)=-s(X+Y)A(X+Y)$$
$$=-s(X)AX-s(X)AY-s(Y)AX-s(Y)AY$$
$$=(\nabla_X JA)X+(\nabla_X JA)Y+(\nabla_Y JA)X+(\nabla_Y JA)Y$$
$$=-s(X)AX-s(Y)AY+(\nabla_X JA)Y+(\nabla_Y JA)X.$$

Hence, by (3.5), we have $(\nabla_X JA)Y=-s(X)AY$

If $X\in T_-(y)$(resp. $T_+(y)$) and $Y\in T_+(y)$(resp. $T_-(y)$) we find

$$(\nabla_X JA)JY=(\nabla_X J)AJY+Js(X)JAJY=(\nabla_X J)AJY-s(X)AJY$$

From $JY\in T_-(y)$(resp. $T_+(y)$)

$$(\nabla_X JA)JY=-s(X)AJY$$

Hence we get $(\nabla_X J)AJY=0$. This is $0=-\lambda(\nabla_X J)JY$(resp. $\lambda(\nabla_X J)JY$).

Therefore we have $(\nabla_X J)AY=\lambda(\nabla_X J)Y$ (resp. $-\lambda(\nabla_X J)Y)=0$.

Hence we see that

$$(\nabla_X JA)Y=(\nabla_X J)AY+J((\nabla_X A)Y)=Js(X)JAY=-s(X)AY.$$

We shll prove the following

〔Theorem 2〕 Let M be a complex hypersurface of a irreducible symmetric K-space $\tilde{M}$. If M is an Einstein then M is locally symmetric.

Proof. It suffices to show that $\nabla R=0$ on U(x), where R is the curvature tensor of M. By virtue of Lemma 3.3,

$$\begin{aligned}\tilde{R}(X,Y,Z,W)&=R(X,Y,Z,W)-\{g(AX,Z)g(AY,W)-g(AX,W)g(AY,Z)\}\\&\qquad-\{g(JAX,Z)g(JAY,W)-g(JAX,W)g(JAY,Z)\}\\&=R(X,Y,Z,W)+D(X,Y,Z,W)\ \text{say,}\end{aligned}$$

where $X,Y,Z,W\in T_y(M)$ and $y\in U(x)$. Considering the tensor field $\tilde{R}$ restricted to M we may write

$$\nabla_V\tilde{R}=\nabla_V R+\nabla_V D$$

where $V\in T_y(M)$. Since $\tilde{M}$ is locally symmetric, we have $\nabla_V\tilde{R}=0$. We know that an irreducible symmetric space is an Einstein space. Hence, by Lemma 3.5 and Theorem 1, we see that

$$\begin{aligned}(\nabla_V D)(X,Y,Z,W)\\&=s(V)\{-g(JAX,Z)g(AY,W)-g(AX,Z)g(JAY,W)\\&\qquad+g(JAX,W)g(AY,Z)+g(AX,W)g(JAY,Z)\\&\qquad+g(AX,Z)g(JAY,W)+g(JAX,Z)g(AY,W)\\&\qquad-g(AX,W)g(JAY,Z)-g(JAX,W)g(AY,Z)\}\\&=0.\end{aligned}$$

Hence $\nabla R=0$ on U(x) or, in other woods, M is locally symmetric.

Bibliography

〔1〕 B. Smyth, Differential geometry of complex hypersurfaces, Ann. of Math. 85 (1967) 246—266.

〔2〕 S. Sawaki & K. Sekigawa, Almost Hermitian manifolds with constant holomorphic sectional curvature, J. Differential Geometry. 9 (1974) 123—134.

昭和51年2月