On Stability and Boundedness of Solutions of a System of Some Functional Differential Equations

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(Received on 31 October 1973)

1. Introduction

In the application, the future behavior of many phenomena is assumed to be descrived by the solutions of an ordinary differential equation. Implicit in this assumption is that the future behavior is uniquely determined by the present, independent of the past. In functional differential equations, the past exerts its influence in a significant manner upon the future. Many models under scrutiny are better represented by functional differential equations than by ordinary differential equations.

In functional differential equations, stability and boundeness are important problems. To apply Lyapunov's second method to functional differential equations, one must actually use a Lyapunov functional. By using Lyapunov functional, one can extend to functional differential equations most of the well-known results for ordinary differential equations. In fact there are many papers concerned with stability and boundeness.

In this paper, we study stability and boundness theorems which use a Lyapunov functional satisfying weaker conditions. The paradigm for this endeavor, of course, is Lyapunov's second method. Before this idea can be made more precise, some basic definitions and notations will be necessary.

2. Definitions and Notations

Suppose $h > 0$ is a given number, $I = [0, \infty)$, R^n is an n-dimensional space, $C([a, b], R^n)$ is the Banach space of continuous functions mapping the interval $[a, b]$ into R^n with the topology of uniform convergence. If $[a, b] = [-h, 0]$, we let $C = C([-h, 0], R^n)$ and designate the norm of an element φ in C by $\|\varphi\| = \sup_{-h \leqq \theta \leqq 0} |\varphi(\theta)|$. If $\sigma \in I$, $A \geqq 0$ and $x \in C([\sigma - h, \sigma + A], R^n)$, then for any $t \in [\sigma, \sigma + A]$ we let $x_t \in C$ be defined by $x_t(\theta) = x(t + \theta)$, $-h \leqq \theta \leqq 0$, i. e., the symbol x_t will denote the restriction of any continuous function $x(u)$ defined on $-h \leqq u < A$, to the interval $[t-h, t]$. C_H will denote the set of $\varphi \in C$ such that $\|\varphi\| \leqq H$.

Let $\dot{x}(t)$ denote the right-hand derivative of $x(u)$ at $u = t$ and consider the functional differential equation of retarded type or simply the functional differential equation

$$\dot{x}(t) = F(t, x_t), \qquad (1)$$

where $F(t, \varphi) \in R^n$ is defined on $I \times C_H$.

Throughout this paper, we assume that for any $\alpha > 0$ there exists an $L(t, \alpha) > 0$ such that if $\|\varphi\| \leqq \alpha$, we have $|F(t, \varphi)| \leqq L(t, \alpha)$, where $L(t, \alpha)$ is continuous in t.

(Definition 1.) A function $x(t_0, \varphi)$ is said to be a solution of (1) with initial condition $\varphi \in C_H$ at $t = t_0$, $t_0 \geqq 0$, if there is an $A > 0$ such that $x(t_0, \varphi)$ is a function from $[t_0 - h, t_0 + A)$ into R^n with the properties;

(i) $x_t(t_0, \varphi) \in C_H$ for $t_0 \leqq t < t_0 + A$,

(ii) $x_t(t_0, \varphi) = \varphi$,

(iii) $x(t_0, \varphi)$ satisfies (1) for $t_0 \leqq t < t_0 + A$.

In this paper, we shall denote by $x(t; t_0, \varphi)$ the value of $x(t_0, \varphi)$ at t.

(Definition 2.) Let $V(t, \varphi)$ be a continuous functional defined for $t \geqq 0$, $\varphi \in C_H$. The upper right-hand derivative of $V(t, \varphi)$ along the solutions of (1) will be denoted by $V'_{(1)}(t, \varphi)$ and is defined to be

$$V'_{(1)}(t, \varphi) = \overline{\lim_{\delta \to 0+}} \frac{1}{\delta} \{ V(t + \delta, x_{t+\delta}(t, \varphi)) - V(t, \varphi) \},$$

where $x(t_0, \varphi)$ is the solution of (1) through (t_0, φ).

(Definition 3.) The zero solution of (1) is said to be stable if for any $\varepsilon > 0$ and $t_0 \in I$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that if $\|\varphi\| < \delta$, we have $\|x_t(t_0, \varphi)\| < \varepsilon$ for all $t \geqq t_0$.

(Definition 4.) The solutions of (1) are equi-bounded, if for any $\alpha > 0$ and $t_0 \in I$, there exists $\beta(t_0, \alpha) > 0$ such that if $\|\varphi\| < \alpha$, $\|x_t(t_0, \varphi)\| < \beta(t_0, \alpha)$ for all $t \geqq t_0$.

(Definition 5.) The solutions of (1) are uniform-bounded, if the β in (De-finition 4.) is independent of t_0.

3. Preliminary Results

In [1], [2] and [3], the sufficient condition for stability of functional differential equation (1) was given by L. E. El'sgol'ts, R. D. Driver and W. Hahn, as follows.

【Theorem 1.】 *The zero solution of the system* (1) *is stable if there exists a continuous functional* $V(t, \varphi)$ *defined on* $I \times C_H$ *which satisfies the following condi-*

tions ;

(i) $V(t, 0) \equiv 0$,

(ii) $a(\|\varphi\|) \leqq V(t, \varphi)$, *where* $a(r)$ *is continuous, positive-definite and monotone-increasing for* $0 \leqq r < \infty$,

(iii) $V'_{(1)}(t, \varphi) \leqq 0$

T. Yoshizawa proved the bounded theorem for the system (1) in [4] and [5].

【Theorem 2.】 *Suppose that there exists a continuous Lyapunov functional* $V(t, \varphi)$ *defined on* $t \in I, \varphi \in S^*$, *where* S^* *denotes the set of* $\varphi \in C$ *such that* $|\varphi(0)| \geqq H$, (H *may be large*), *which satisfies the following conditions*;

(i) $a(|\varphi(0)|) \leqq V(t, \varphi) \leqq b_1(|\varphi(0)|) + b_2(\|\varphi\|)$, *where* $a(r)$, $b_1(r)$, $b_2(r) \in CI$, *positive for* $r > H$ *and* $a(r) - b_2(r) \to \infty$ *as* $r \to \infty$,

(ii) $V'_{(1)}(t, \varphi) \leqq 0$.

Then, the solutions of (1) *are uniform-bounded.*

For proofs of these theorems, see references.

4. Stability

Consider the system of functional differential equation (1) and suppose that $F(t, \varphi)$ is defined and continuous on $I \times C_H$ and that $F(t, 0) \equiv 0$.

【Theorem 3.】 *Suppose that there exists a continuous Lyapunov functional* $V(t, \varphi)$ *defined on* $I \times C_H$ *which satisfies the following conditions* ;

(i) $V(t, 0) \equiv 0$,

(ii) $a(t, \|\varphi\|) \leqq V(t, \varphi)$, *where* $a(t, r)$ *is continuous in* (t, r) *on* $I \times C_H$, $a(t, 0) \equiv 0$, $a(t, r) > 0$ *for* $r \neq 0$ *and increases monotonically with respect to* t *for each fixed* r,

(iii) $V'_{(1)}(t, \varphi) \leqq 0$.

Then the zero solution of the system (1) *is stable.*

Proof. For any $\varepsilon > 0$, there exists a $\delta(t_0, \varepsilon) > 0$ such that $\|\varphi\| < \delta(t_0, \varepsilon)$ implies $V(t_0, \varphi) < a(t_0, \varepsilon)$, because $V(t, 0) \equiv 0$ and $V(t, \varphi)$ is continuous.

Suppose that there exists a $t_0 < t_1 < \infty$ such that $\|x_{t_1}(t_0, \varphi)\| = \varepsilon$ and $\|x_t(t_0, \varphi)\| < \varepsilon$ for $t \in [t_0, t_1)$. By (iii),

$a(t_1, \varepsilon) = a(t_1, \|x_{t_1}(t_0, \varphi)\|) \leqq V(t_1, x_{t_1}(t_0, \varphi)) \leqq V(t_0, \varphi) < a(t_0, \varepsilon) < a(t_1, \varepsilon)$.

This is a contradiction, and hence, if $\|\varphi\| < \delta(t_0, \varepsilon)$, then $\|x_t(t_0, \varphi)\| < \varepsilon$ for all $t \geqq t_0$.

5. Boundedness

In this section we consider the system (1), where $F(t, \varphi)$ of (1) is defined and continuous on $I \times C$.

【Theorem 4.】 *Suppose that there exists a continuous Lyapunov functional* $V(t, \varphi)$ *defined on* $I \times S$, *where* S *is the set of* $\varphi \in C$ *such that* $\|\varphi\| \geqq H$ ($H > 0$ *may be large*), *which satisfies the following conditions* ;

(i) $a(t, \|\varphi\|) \leqq V(t, \varphi)$, *where* $a(t, r)$ *is continuous in* (t, r) *and* $a(t, r) \to \infty$ *uniformly in* t *as* $r \to \infty$,

(ii) $V'_{(1)}(t, \varphi) \leqq 0$.

Then, the solutions of the system (1) *are equi-bounded.*

Proof. Let $x(t_0, \varphi)$ be a solution of (1) such that $t_0 \in I$ and $\varphi \in C_\alpha$ for any $\alpha > 0$. By continuity of the functional $V(t, \varphi)$, there is a positive constant $K(t_0, \alpha)$ such that if $\varphi \in C_\alpha$, $V(t_0, \varphi) \leqq K(t_0, \alpha)$. By condition (i), there exists a constant $\beta(t_0, \alpha) > 0$ so large that $K(t_0, \alpha) < a(t, \beta(t_0, \alpha))$ for any $t \geqq t_0$. Suppose that $\|x_{t_1}(t_0, \varphi)\| = \beta(t_0, \alpha)$ at some t_1, $t_1 > t_0$. By (ii), $V(t_1, x_{t_1}(t_0, \varphi)) \leqq V(t_0, x_{t_0}(t_0, \varphi)) = V(t_0, \varphi) \leqq K(t_0, \alpha)$, which implies $a(t_1, \beta(t_0, \alpha)) \leqq K(t_0, \alpha)$. This contradicts the choice of $\beta(t_0, \alpha)$. This completes the proof.

【Theorem 5.】 *Suppose that there exists a continuous Lyapunov functional* $V(t, \varphi)$ *defined on* $I \times S$ *satisfying the following conditions*;

(i) $a(t, \|\varphi\|) \leqq V(t, \varphi) \leqq b(\|\varphi\|)$, *where* $a(t, r)$ *is continuous in* (t, r), $a(t, r) \to \infty$ *uniformly in* t *as* $r \to \infty$ *and* $b(r)$ *is continuous*,

(ii) $V'_{(1)}(t, \varphi) \leqq 0$.

Then the solutions of the system (1) *are uniformly bounded.*

Proof. Since a function $b(r)$ is continuous, there exists a constant $B(\alpha) > 0$ such that $b(\|\varphi\|) \leqq B(\alpha)$ for $H \leqq \|\varphi\| \leqq \alpha$. By condition (i), $V(t, \varphi) \leqq B(\alpha)$ for $H \leqq \|\varphi\| \leqq \alpha$, we can choose a $\beta(\alpha) > 0$ so large that $B(\alpha) < a(t, \beta(\alpha))$ for all $t \geqq t_0$. Suppose that $\|x_{t_1}(t_0, \varphi)\| = \beta(\alpha)$ at some t_1, where $x(t_0, \varphi)$ is a solution of system (1) through (t_0, φ). Then there exist t_2 and t_3, $t_0 \leqq t_2 < t_3 \leqq t_1$, such that $\|x_{t_2}(t_0, \varphi)\| = \alpha$, $|x(t_3; t_0, \varphi)| = \beta(\alpha)$ and $\alpha < \|x_t(t_0, \varphi)\| < \beta(\alpha)$ for all $t \in (t_2, t_3)$. In virtue of condition (ii), $V(t, \varphi)$ is nonincreasing along the solutions of (1). Therefore,

$$V(t_3, x_{t_3}(t_0, \varphi)) \leqq V(t_2, x_{t_2}(t_0, \varphi)) \leqq b(\|x_t\|) \leqq B(\alpha),$$

which implies $a(t_3, \beta(\alpha)) \leqq B(\alpha)$. This is a contradiction, and the theorem is proved.

【Theorem 6.】 *Suppose that there exists a contiuous Lyapunov functional* $V(t, \varphi)$ *defined on* $I \times S^*$ *which satisfies the following conditions*;

(i) $a(t, |\varphi(0)|) \leqq V(t, \varphi) \leqq b_1(|\varphi(0)|) + b_2(t, \|\varphi\|)$, *where* $a(t, r)$ *and* $b_2(t, r)$ *are continuous in* (t, r), *positive for* $r > H$ *and*

$$a(t, r) - b_2(t, r) \to \infty$$

uniformly in t *as* $r \to \infty$, $a(t, r)$ *increases monotonically with respect to* t *for any fixed* r, $b_2(t, r)$ *is increasing function with respect to* r *for any fixed* t *and* $b_1(r)$ *is continuous, positive for* $r > H$ *and increasing*,

(ii) $V'_{(1)}(t, \varphi) \leqq 0$.

Then the solutions of the system (1) *are uniformly bounded.*

Proof. For a given $\alpha > H$, choose $\beta(\alpha) > 0$ so large that

$$b_1(\alpha) + b_2(t, \beta(\alpha)) < a(t, \beta(\alpha)).$$

Let $x(t_0, \varphi)$ be a solution of (1) such that $t_0 \in I$ and $\varphi \in C$. Suppose there exists $t_1, t_1 > t_0$, such that $\|x_{t_1}(t_0, \varphi)\| = \beta(\alpha)$. Then there exist t_2 and t_3, $t_0 \leqq t_2 < t_3 \leqq t_1$, such that $|x(t_2; t_0, \varphi)| = \alpha$, $|x(t_3; t_0, \varphi)| = \beta(\alpha)$ and that $\alpha < |x(t; t_0, \varphi)| < \beta(\alpha)$ for all $t \in (t_2, t_3)$, and we can assume that $|x(t; t_0, \varphi)| < \beta(\alpha)$ for all $t \in [t_0, t_3)$, and hence $\|x_{t_2}(t_0, \varphi)\| < \beta(\alpha)$. For $t \in [t_2, t_3]$, $x_t(t_0, \varphi) \in S^*$ and hence, by (ii),
$V(t_3, x_{t_3}(t_0, \varphi)) \leqq V(t_2, x_{t_2}(t_0, \varphi))$. This implies

$$a(t_2, \beta(\alpha)) \leqq b_1(\alpha) + b_2(t_2, \beta(\alpha)).$$

Thus, $\|x_t(t_0, \varphi)\| < \beta(\alpha)$ for all $t \geqq t_0$, this completes the proof of the theorem.

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