On Stability and Boundedness of Solutions of a System of Ordinary Differential Equations

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1 Introduction

Lyapunov's second method is a useful approach to the study of stability and boundedness of solutions of ordinary differential equations. Roughly speaking, his method involves the use of a real nonnegative function V(t, x), where t is the independent variable and x is the dependent variable. However, it is difficult to find the function V(t, x) satisfying certain conditions. Therefore, it is important to obtain a weak sufficient condition for stability and boundedness theorem. In the previous paper [3], we obtained a weak sufficient condition for a stability theorem.

The purpose of this paper is to give some sufficient condition for Lyapunov's stability and boundedness theorem.

2 Definitions and Notations

Let I denote the interval $0 \leqq t < \infty$ and R^n denote Euclidean n-space.

For $x \in R^n$, let $\|x\|$ be the Euclidean norm of x, and D is a domain such that $\|x\| \leqq H$, $H > 0$.

We shall sometimes denote by S_α the set of x such that $\|x\| \leqq \alpha$.

We consider a system of differential equations

$$\frac{dx}{dt} = f(t, x), \tag{1}$$

where x is an n-dimensional vector f(t, x) is an n-dimensional vector function which is defined on a region in $I \times R^n$, and is continuous in (t, x) on $I \times D$.

Throughout this paper a solution through a point (t_0, x_0) in $I \times R^n$ will be denoted by such a form as $x(t; x_0, t_0)$.

We introduce the following definitions.

Definition 1. The equilibrium of the system (1) is said to be stable if for any $\varepsilon > 0$ and any $t_0 \in I$ there exists a $\delta(t_0, \varepsilon) > 0$ such that the inequality $\|x_0\| < \delta$ implies $\|x(t; x_0, t_0)\| < \varepsilon$ for all $t \geqq t_0$.

Definition 2. The equilibrium of the system (1) is said to be asymptotically stable if it is stable and if there exists a $\delta_0(t_0)>0$ such that if $\|x_0\|<\delta_0(t_0)$, $x(t;x_0,t_0)\to 0$ as $t\to\infty$.

Definition 3. The equilibrium of the system (1) is said to be quasi-equi-asymptotically stable in the large if for any $\alpha>0$, any $\varepsilon>0$ and $t_0\in I$, there exists a $T(t_0,\varepsilon,\alpha)>0$ such that if $x_0\in S_\alpha$, $\|x(t;x_0,t_0)\|<\varepsilon$ for all $t\geqq t_0+T(t_0,\varepsilon,\alpha)$.

Definition 4. The equilibrium of the system (1) is said to be equiasymptotically stable in the large if it is stable and is quasi-equi-asymptotically stable in the large.

Definition 5. The solutions of the system (1) are equi-bounded, if for any $\alpha>0$ and $t_0\in I$, there exists a $\beta(t_0,\alpha)>0$ such that if $x_0\in S_\alpha$, $\|x(t;x_0,t_0)\|<\beta(t_0,\alpha)$ for all $t\geqq t_0$.

Definition 6. The solutions of the system (1) are uniform-bounded, if the β in the above Definition 5 is independent of t_0.

Definition 7. The solutions of the system (1) are equiultimately bounded for bound B, if there exists a $B>0$ and if corresponding to any $\alpha>0$ and $t_0\in I$, there exists a $T(t_0,\alpha)>0$ such that $x_0\in S_\alpha$ implies that $\|x(t;x_0,t_0)\|<B$ for all $t\geqq t_0+T(t_0,\alpha)$.

Definition 8. Let $V(t,x)$ be a continuous scalar function defined on an open set, and which satisfies locally a Lipschitz condition with respect to x. Corresponding to $V(t,x)$, we define the function

$$V'_{(1)}(t,x)=\overline{\lim_{h\to 0+}}\frac{1}{h}\{V(t+h,x+hf(t,x))-V(t,x)\}.$$

In case $V(t,x)$ has continuous partial derivatives of the first order, it is evident that

$$V'_{(1)}(t,x)=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}\cdot f(t,x),$$

where " $\cdot$ " denotes a scalar product.

3 Preliminary Results

【Theorem 1】 *Suppose that there exists a Lyapunov function* $V(t,x)$ *defined on* $I\times D$, *which satisfies the following conditions;*

(i) $V(t,0)\equiv 0$ *and* $V(t,x)$ *is continuous in* (t,x),

(ii) $a(t,\|x\|)\leqq V(t,x)$, *where the function* $a(t,r)$ *is continuous in* (t,r) *on* $I\times D$, $a(t,0)\equiv 0$, $a(t,r)>0$ *for* $r\neq 0$ *and increases monotonically with respect to t for each fixed* r,

(iii) $V'_{(1)}(t,x)\leqq 0$.

Then the equilibrium of the system (1) *is stable.*

For proof of this theorem, see reference [3].

【Theorem 2】 *Suppose that the maximal solution* $u(t)$ *of a scalar differential equation*

$\frac{du}{dt} = g(t, u)$, *where* $g(t, u)$ *is continuous on* $E: 0 \leqq t \leqq T$, $|u| < A$, $A > 0$ *such that* $u(t_0) = u_0$, $(t_0, u_0) \in E$, *stays in* E *for* $t \in [t_0, T]$. *If a continuous function* $y(t)$ *with* $y(t_0) = u_0$ *satisfies*

$$y'(t) = \overline{\lim_{h \to 0+}} \frac{1}{h} \{y(t+h) - y(t)\} \leqq g(t, y(t))$$

on $[t_0, T]$, $|u| < A$, *then we have* $y(t) \leqq u(t)$ *for* $t \in [t_0, T]$.

For proof of this theorem, see reference [1].

4 Stability

[Theorem 3] *Suppose that* $f(t, x)$ *is bounded*, $f(t, 0) \equiv 0$ *and that there exists a Lyapunov function* $V(t, x)$ *defined on* $I \times D$, *which satisfies the following conditions;*

(i) $V(t, 0) \equiv 0$ *and* $V(t, x)$ *is continuous in* (t, x),

(ii) $a(t, \|x\|) \leqq V(t, x)$, *where the function* $a(t, r)$ *is continuous in* (t, r) *on* $I \times D$, $a(t, 0) \equiv 0$, $a(t, r) > 0$ *for* $r \neq 0$ *and increases monotonically with respect to* t *for each fixed* r,

(iii) $V'_{(1)}(t, x) \leqq -c(t, \|x\|)$, *where the function* $c(t, r)$ *is continuous in* (t, r) *on* $I \times D$, $c(t, 0) \equiv 0$ *and* $c(t, r) > 0$ *for* $r \neq 0$.

Then the equilibrium of the system (1) *is asymptotically stable.*

Proof. By Theorem 1, the equilibrium of the system (1) is stable. Suppose the equilibrium of the system (1) is not asymptotically stable. Then for some $\varepsilon > 0$ there exists a solution $x(t; x_0, t_0)$ and a divergent sequence $\{t_k\}$ for which $\|x(t_k; x_0, t_0)\| \geqq \varepsilon$. Since $f(t, x)$ is bounded, there exists a $K > 0$ such that $\left|\frac{d\|x\|}{dt}\right| < K$. Therefore, on the intervals

$$t_k - \frac{\varepsilon}{2K} \leqq t \leqq t_k + \frac{\varepsilon}{2K}, \tag{2}$$

we heve $\|x(t; x_0, t_0)\| \geqq \frac{\varepsilon}{2}$. We can assume that these intervals are disjoint and $t_1 - \frac{\varepsilon}{2K} > t_0$ by taking, if necessary, a subsequence of $\{t_k\}$.

Since $V'_{(1)}(t, x) \leqq -c(t, \|x\|)$, there exists a constant $\gamma > 0$ such that $V'_{(1)}(t, x) \leqq -\gamma$ on the intervals (2), and $V'_{(1)}(t, x) \leqq 0$ elsewhere. Therefore,

$$V\left(t_k + \frac{\varepsilon}{2K}, x\left(t_k + \frac{\varepsilon}{2K}; x_0, t_0\right)\right) - V(t_0, x_0) < -\gamma \frac{\varepsilon}{K} k \to -\infty$$

as $k \to \infty$, which contradicts $V(t, x) \geqq 0$. Thus, we see that the equilibrium of the system (1) is asymptotically stable.

[Theorem 4] *Suppose that* $f(t, 0) \equiv 0$ *and that there exists a Lyapunov function* $V(t, x)$ *defined on* $I \times D$, *which satisfies the following conditions;*

(i) $V(t, 0) \equiv 0$ *and* $V(t, x)$ *is continuous in* (t, x),

(ii) $a(t, \|x\|) \leqq V(t, x)$, *where the function* $a(t, r)$ *is continuous in* (t, r) *on* $I \times D$, $a(t, 0) \equiv 0$, $a(t, r) > 0$ *for* $r \neq 0$ *and increases monotonically* *with respect to* t *for each fixed* r, *and* $a(t, s(t)) \to 0$ *implies* $s(t) \to 0$ *as* $t \to \infty$,

(iii) $V'_{(1)}(t, x) \leqq -c(V(t, x))$, *where the function* $c(r)$ *is continuous and monotone-increasing and* $c(0) = 0$.

Then the equilibrium of the system (1) *is asymptotically stable.*

Proof. By Theorem 1, the equilibrium of the system (1) is stable. By (iii), $V(t, x(t; x_0, t_0))$ is monotone-decreasing; hence the limit

$V_0 = \lim_{t\to\infty} V(t, x(t; x_0, t_0))$ exists.

If $V_0 \neq 0$, we have $c(V_0) \neq 0$, and since $c(r)$ is monotone-increasing,

$$c(V(t, x(t; x_0, t_0))) > c(V_0).$$

Hence $V'_{(1)}(t, x) < -c(V_0)$. Integrating, we have

$$V(t, x(t; x_0, t_0)) - V(t_0, x_0) \leqq -c(V_0)(t - t_0).$$

Thus $V(t, x(t; x_0, t_0))$ diverges to $-\infty$ as $t \to \infty$, which contradicts the fact that $V(t, x(t; x_0, t_0)) \geqq a(t, \|x(t; x_0, t_0)\|)$.

It follows that $V_0 = 0$; from $V(t, x(t; x_0, t_0)) \to 0$, it follows that $a(t, \|x(t; x_0, t_0)\|) \to 0$, and thus that $x(t; x_0, t_0) \to 0$ when $t \to \infty$. This proves the theorem.

【Theorem 5】 *Suppose that* $f(t, 0) \equiv 0$ *and that there exists a Lyapvnov function* $V(t, x)$ *defined on* $I \times R^n$ *which satisfies the following conditions*;

(i) $V(t, 0) \equiv 0$ *and* $V(t, x)$ *is continuons in* (t, x),

(ii) $a(t, \|x\|) \leqq V(t, x)$, *where the function* $a(t, r)$ *is continuous in* (t, r), $a(t, 0) \equiv 0$, $a(t, r) > 0$ *for* $r \neq 0$ *and increases monotonically with respect to* t *for each fixed* r, *and* $a(t, r) \to \infty$ *uniformly in* t *as* $r \to \infty$,

(iii) $V'_{(1)}(t, x) \leqq -cV(t, x)$, *where* $c > 0$ *is a constant.*

Then the equilibrium of the system (1) *is equiasymptotically stable in the large.*

Proof. By Theorem 1, the equilibrium of the system (1) is stable. Moreover, the solutions of (1) are equi-bounded. This fact will be proved later (cf. §5, Theorem 7). Hence, every solution exists in the future. Let $x(t; x_0, t_0)$ be a solution such that $\|x_0\| \leqq \alpha$. Applying Theorem 2, by (iii)

$$V(t, x(t; x_0, t_0)) \leqq V(t_0, x_0)e^{-c(t-t_0)}.$$

Let $M(t_0, \alpha) = \max_{\|x_0\| \leqq \alpha} V(t_0, x_0)$, and let $T(t_0, \varepsilon, \alpha)$ be such that $\|x_0\| \leqq \alpha$

$$T(t_0, \varepsilon, \alpha) = \frac{1}{c} \log \frac{M(t_0, \alpha)}{a(t_0, \varepsilon)}.$$

Suppose at some $t_1 > t_0 + T$, $\|x(t_1; x_0, t_0)\| = \varepsilon$. Then

$$a(t_1, \varepsilon) \leqq V(t_1, x(t_1; x_0, t_0)) \leqq V(t_0, x_0)e^{-c(t_1-t_0)} < V(t_0, x_0)e^{-cT} \leqq M(t_0, \alpha)e^{-\log\frac{M(t_0, \alpha)}{a(t_0, \varepsilon)}}$$
$$= a(t_0, \varepsilon).$$

This contradicts the condition (ii), and hence, if $\|x_0\| \leqq \alpha$, $\|x(t; x_0, t_0)\| < \varepsilon$ for all $t \geqq t_0 + T(t_0, \varepsilon, \alpha)$, that is, the equilibrium of the system (1) is quasi-equiasymptotically stable in the large. This completes the proof.

【Theorem 6】 *Suppose that there exists a Lyapunov function* $V(t, x)$ *defined on* $I \times R^n$ *such that* $a(t, \|x\|) \leqq V(t, x)$, *where the function* $a(t, r)$ *is continuous* *in* (t, r), $a(t, 0) \equiv 0$, $a(t, r) > 0$ *for* $r \neq 0$ *and* $a(t, s(t)) \to 0$ *implies* $s(t) \to 0$ *as* $t \to \infty$. *Moreover, suppose that there exists a continuous scalar function* $w(t, r)$ *defined on* $0 \leqq t < \infty$, $r \geqq 0$ *such that*

$$V'_{(1)}(t, x) \leqq w(t, V(t, x)).$$

Then, if all solutions of

$$\frac{dr}{dt} = w(t, r), \tag{3}$$

tend to zero as $t \to \infty$, *all solutions of* (1) *tend to zero.*

Proof. Let $x(t\,;x_0, t_0)$ be a solution of (1) and let $r(t)$ be the maximal solution of (3) such that $r(t_0) = V(t_0, x_0)$. Since $r(t) \to 0$ as $t \to \infty$, $V(t, x(t; x_0, t_0)) \to 0$ as $t \to \infty$, which implies that $x(t\,; x_0, t_0) \to 0$ as $t \to \infty$.

5 Boundedness

【Theorem 7】 *Suppose that there exists a Lyapunov function* $V(t, x)$ *defined on* $I \times R^n$ *which satisfies the following conditions ;*

(i) $a(t, \|x\|) \leqq V(t, x)$, *where the function* $a(t, r)$ *is continuous in* (t, r) *and* $a(t, r) \to \infty$ *uniformly in* t *as* $r \to \infty$,

(ii) $V'_{(1)}(t, x) \leqq 0$.

Then the solutions of the system (1) *are equi-bounded.*

Proof. For any given $\alpha > 0$, let $x(t\,; x_0, t_0)$ be a solution of (1) such that $t_0 \in I$ and $x_0 \in S_\alpha$. Since $V(t, x)$ is continuous, there exists a $K(t_0, \alpha) > 0$ such that if $x_0 \in S_\alpha$; $V(t_0, x_0) \leqq K(t_0, \alpha)$. By (i), we can choose a $\beta(t_0, \alpha) > 0$ so large that $a(t, \beta(t_0, \alpha)) > K(t_0, \alpha)$ for any $t \geqq t_0$. Suppose that $\|x(t_1\,; x_0, t_0)\| = \beta(t_0, \alpha)$ at some t_1, $t_1 > t_0$, By (ii),

$$V(t_1, x(t_1; x_0, t_0)) \leqq V(t_0, x_0),$$

which implies $a(t_1, \beta(t_0, \alpha)) \leqq K(t_0, \alpha)$. This contradicts the choice of $\beta(t_0, \alpha)$. Thus $\|x(t\,; x_0, t_0)\| < \beta(t_0, \alpha)$ for all $t \geqq t_0$. This shows the equi-boundedness of solutions of (1).

【Theorem 8】 *Suppose that there exists a Lyapunov function* $V(t, x)$ *defined on* $0 \leqq t < \infty$, $\|x\| \geqq R$ *where R may be large, which satisfies the following conditions;*

(i) $a(t, \|x\|) \leqq V(t, x) \leqq b(\|x\|)$, *where the function* $a(t, r)$ *is continuous in* (t, r) *and* $a(t, r) \to \infty$ *uniformly in* t *as* $r \to \infty$, *and the function* $b(r)$ *is continuous,*

(ii) $V'_{(1)}(t, x) \leqq 0$.

Then the solutions of the system (1) *are uniform-bounded.*

Proof. By continuity of the function $b(r)$, there exists a $B(\alpha) > 0$ such that $b(\|x\|) \leqq B(\alpha)$ for $R \leqq \|x\| \leqq \alpha$. By (i), $V(t, x) \leqq B(\alpha)$ for $R \leqq \|x\| \leqq \alpha$, and we can choose a $\beta(\alpha) > 0$ so large that $a(t, \beta(\alpha)) > B(\alpha)$ for any $t \geqq t_0$. Suppose that $\|x(t\,; x_0, t_0)\| = \beta(\alpha)$ at some t. Then there exist t_1 and t_2, $t_0 \leqq t_1 < t_2$, such that $\|x(t_1\,; x_0, t_0)\| = \alpha$, $\|x(t_2\,; x_0, t_0)\| = \beta(\alpha)$ and that $\alpha < \|x(t\,; x_0, t_0)\| < \beta(\alpha)$ for $t \in (t_1, t_2)$. By (i) and (ii),

$$a(t_2, \beta(\alpha)) = a(t_2, \|x(t_2; x_0, t_0)\|) \leqq V(t_2, x(t_2; x_0, t_0))$$
$$\leqq V(t_1, x(t_1; x_0, t_0)) \leqq b(\alpha)$$

which implies $a(t_2, \beta(\alpha)) < B(\alpha)$. This contradicts the choice of $\beta(\alpha)$. Thus $\|x(t; x_0, t_0)\| < \beta(\alpha)$ for all $t \geqq t_0$. This shows the uniform-boundedness of solutions of (1).

【Theorem 9】 *Suppose that there exists a Lyapunov function* $V(t, x)$ *defined on* $I \times R^n$, *which satisfies the following conditions ;*

(i) $a(t, \|x\|) \leqq V(t, x)$ *for* $\|x\| \geqq B$, *where the function* $a(t, r)$ *is continuous in* (t, r) *and monotone-increasing with respect to* t *for each fixed* r *and to* r *for each fixed* t, *and* $a(t, r) \to \infty$ *uniformly in* t *as* $r \to \infty$,

(ii) $V'_{(1)}(t, x) \leqq -cV(t, x)$, *where* $c > 0$ *is a constant.*

Then the solutions of the system (1) *are equiultimately bouuded for bound* B.

Proof. Since $V(t, x)$ is continuous, there exists a $K(t_0, \alpha) > 0$ such that if $x_0 \in S_\alpha$, $V(t_0, x_0) \leqq K(t_0, \alpha)$. Let $x(t; x_0, t_0)$ be a solution of (1) such that $x_0 \in S_\alpha$. It is bounded for all $t \geqq t_0$. Suppose that there exists some

$t_1 > t_0 + \dfrac{1}{c} \log \dfrac{K(t_0, \alpha)}{a(t_0, B)}$ such that $\|x(t_1; x_0, t_0)\| \geqq B$. From (i) and (ii),

$$a(t_0, B) \leqq a(t_1, \|x(t_1; x_0, t_0)\|) \leqq V(t_1, x(t_1; x_0, t_0)) \leqq V(t_0, x_0)e^{-c(t-t_0)}$$
$$< K(t_0, \alpha)\, e^{-\log\frac{K(t_0, \alpha)}{a(t_0, B)}} = a(t_0, B).$$

This is a contradiction. Therefore, if $t > t_0 + \dfrac{1}{c} \log \dfrac{K(t_0, \alpha)}{a(t_0, B)}$, we have $\|x(t; x_0, t_0)\| < B$. Thus, the solutions of (1) are equiultimately bounded for bound B.

References

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