# Application of Finite Element Method for Eigenvalue Problem

Takashi Yoshimura

(Received on 31 October, 1973)

## 1. Approximation procedure[1),2)]

Let R be a region with boundary $\partial R$ and consider the second order steady state problem

(1) $\qquad -\nabla(p\nabla u) + qu = f \qquad$ in R

(2) $\qquad u = 0 \qquad$ on $\partial R$

Let $a(u, v)$ denote the bilinear form associated with the second order operator, i.e.,

$$a(u, v) = \int_R (p\nabla u\nabla v + quv)\,dx$$

for u, v in the Sobolev space $H = H_0^1(R)$, and $b(u, v)$ denote the bilinear form

$$b(u, v) = \int_R uv\, dx.$$

The weak solution form of (1) − (2) is to seek a $u_0 \in H$ such that

(3) $\qquad a(u_0, v) = b(f, v) \qquad \forall v \in H.$

From (3), integrating by parts and with (2), it follows that

$$\int_R (-\nabla(p\nabla u) + qu - f)\, v\, dx = 0 \qquad \forall v \in H.$$

Since v is an arbitrary element of H, it follows that $-\nabla(p\nabla u) + qu - f$ is orthogonal to H.

To approximate this problem we first subdivide R into triangles (or rectangles) $e_\alpha$.

We then choose a subspace $S = S(h) \subset H$ of continuous functions which are polynomials of degree k—1 over each element $e_\alpha$. An approximate solution $\bar{u} \in S$ is determined from

$$a(\bar{u}, \bar{v}) = b(f, \bar{v}) \qquad \forall \bar{v} \in S.$$

Letting $\phi_1, \cdots\cdots, \phi_N$ be a basis for S, we construct an approximation in the form

(4) $\qquad \bar{u} = \sum_{i=1}^{N} u_i \phi_i.$

If the finite element approximation $\bar{u}$ of the form (4) only approximately satisfy (1), then the residual

$$r(x) = -\nabla(p\nabla\bar{u}) + q\bar{u} - f$$

will not necessarily be orthogonal to H.

However, in the clasical approximation theory it is well known that the following projection theorem holds: Let H be a Hilbert space and S a closed subspace of H. Then for every $u \in H$ there exists a unique element $\bar{u}^* \in S$ such that $\|u - \bar{u}^*\| < \|u - \bar{u}\|$ for all $\bar{u} \in S$, and the distance $\|u - \bar{u}^*\|$ is minimum if and only if $u - \bar{u}^*$ is orthogonal to S.

Now, we select the coefficients $u_i$ of the approximation so that $r(x)$ is orthogonal to the finite dimensional subspace S spanned by the interpolating functions $\phi_i(x)$.

This amounts to setting

$$\int_R (-\nabla(p\nabla\bar{u}) + q\bar{u} - f)\bar{v}\,dx = 0 .$$

However, since $\bar{v}$ is now an arbitrary element of S, it can be expressed in the form $\bar{v} = \sum_i v_i \phi_i(x)$. Thus it follows that

$$(5) \qquad v_i \int_R (-\nabla(p\nabla\bar{u}) + (q\bar{u}-f)\phi_i\,dx = 0 .$$

But (5) must hold for arbitrary coefficients $v_i$. Therefore

$$\int_R r(x)\phi_i(x)\,dx = \int_R (-\nabla(p\nabla\bar{u}) + q\bar{u} - f)\phi_i\,dx = 0 .$$

This is equivalent to matrix problem

$$KU = F$$

in the vector U of weights $u_i$.

The (i, j)-th entry of the Stiffness matrix K is

$$\int_R (p\nabla\phi_i \nabla\phi_j + q\phi_i\phi_j)\,dx$$

and the j-th entry of the source vector F is

$$\int_R f\phi_j\,dx .$$

Under the usual regurarity assumptions, the error $u-\bar{u}$ is of order $O(h^k)$.

In what follows we assume that our bilinear form depends on parameter $\lambda$. That is our problem in general is to find the value $\lambda=\lambda_0$, such that there exists nontrivial $u_0 \in H$, with

$$(6) \qquad a(u_0, v) = \lambda_0 b(u_0, v) \qquad \forall v \in H,$$

where $a(u, v)$ and $b(u, v)$ be bilinear forms defined on H.

To approximate the problem we introduce a finite dimensional subspace $S \subset H$, parameterized by $h > 0$, and solve (6) in S. That is we seek those complex numbers $\lambda_0$ such that there is a nontrivial $\bar{u} \in S$ satisfying the equation

$$(7) \qquad a(\bar{u}, \bar{v}) = \lambda_0 b(\bar{u}, \bar{v}) \qquad \forall \bar{v} \in S.$$

Let $\bar{u}(x) = \sum_{i=1}^{N} u_i \phi_i(x)$ be a finite element approximation of u. Then (7) is equivalent to the matrix problem of the form

$$KU = \lambda MU$$

The (i, j)-th entry of the stiffness matrix K is $a(\phi_i, \phi_j)$ and the (i, j)-th entry of the mass matrix M is $b(\phi_i, \phi_j)$.

## 2. Finite elements and integral formulas

(i) Triangular element[3]

Consider triangle with nodes 1, 2, 3 whose Cartesian co-ordinates are $(x_i, y_i)$ $i = 1, 2, 3$.

We introduce basis functions such that

$$\phi_i = \frac{1}{2A}(a_i + b_i x + c_i y) \qquad i = 1, 2, 3$$

where A denotes area of triangle 1 2 3, and

$$a_1 = x_2 y_3 - x_3 y_2$$
$$b_1 = y_2 - y_3$$
$$c_1 = x_3 - x_2 \qquad \text{etc.}$$

Then we obtain the following integral formulas.

$$\frac{1}{A}\iint \phi_i \phi_j \, dxdy = \begin{cases} 1/12 & i \neq j \\ 2/12 & i = j \end{cases}$$

$$\frac{1}{A}\iint \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} dxdy = b_i b_j$$

$$\frac{1}{A}\iint \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} dxdy = c_i c_j$$

(ii) Rectangular element[4]

Consider a rectangle with sides parallel to co-ordinate axes, with nodes numbered 1 to 4, whose co-ordinates are $(x_i, y_j)$, $(x_{i+1}, y_j)$, $(x_{i+1}, y_{j+1})$, $(x_i, y_{j+1})$ respectively.

For the basis functions we introduce a piecewise bilinear polynomials as

$$\phi_1(x, y) = l_i(x) l_j(y)$$
$$\phi_2(x, y) = l_{i+1}(x) l_j(y)$$
$$\phi_3(x, y) = l_{i+1}(x) l_{j+1}(y)$$
$$\phi_4(x, y) = l_i(x) l_{j+1}(y)$$

$$\text{where } \; l_i(x) = \begin{cases} (x - x_{i-1})(x_i - x_{i-1})^{-1} & x_{i-1} \leq x \leq x_i \\ (x_{i+1} - x)(x_{i+1} - x_i)^{-1} & x_i \leq x \leq x_{i+1} \\ 0 & x \leq x_{i-1} \text{ or } x_{i+1} \leq x. \end{cases}$$

Then the element matrices are

$$\left[\iint \phi_i \phi_j \, dxdy\right] = \frac{(x_{i+1} - x_i)(y_{j+1} - y_j)}{36} \begin{pmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{pmatrix}$$

$$\left[\iint \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} dxdy\right] = \frac{(y_{j+1} - y_j)}{6(x_{i+1} - x_i)} \begin{pmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{pmatrix}$$

$$\left[\iint \frac{\partial\phi_i}{\partial y}\cdot\frac{\partial\phi_j}{\partial y}\,dxdy\right] = \frac{(x_{i+1}-x_i)}{6(y_{j+1}-y_j)}\begin{pmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{pmatrix}$$

(iii) For the quadratic polynomial basis function in a triangular element we have following integral formulas

$$\frac{1}{A}\iint \phi_i\,\phi_j\,dxdy = \frac{1}{180}\begin{pmatrix} 6 & -1 & -1 & -4 & 0 & 0 \\ -1 & 6 & -1 & 0 & -4 & 0 \\ -1 & -1 & 6 & 0 & 0 & -4 \\ -4 & 0 & 0 & 32 & 16 & 16 \\ 0 & -4 & 0 & 16 & 32 & 16 \\ 0 & 0 & -4 & 16 & 16 & 32 \end{pmatrix}$$

$$\frac{1}{A}\iint\left(\frac{\partial\phi_i}{\partial x}\cdot\frac{\partial\phi_j}{\partial x}+\frac{\partial\phi_i}{\partial y}\cdot\frac{\partial\phi_j}{\partial y}\right)dxdy = \begin{pmatrix} 12d_1 & -4e_3 & -4e_2 & 0 & 16e_2 & 16e_3 \\ -4e_3 & 12d_2 & -4e_1 & 16e_1 & 0 & 16e_3 \\ -4e_2 & -4e_1 & 12d_3 & 16e_1 & 16e_2 & 0 \\ 0 & 16e_1 & 16e_1 & f & 32e_3 & 32e_2 \\ 16e_2 & 0 & 16e_2 & 32e_3 & f & 32e_1 \\ 16e_3 & 16e_3 & 0 & 32e_2 & 32e_1 & f \end{pmatrix}$$

where $d_j = b_j^2 + c_j^2$, $e_i = b_j b_k + c_j c_k$, $f = 16(d_1 + d_2 + d_3)$

## 3. Numerical example

Consider a simplest eigenvalue problem

$$\nabla^2 u + \lambda u = 0 \qquad \text{in unit square}$$

$$u = 0 \qquad \text{on the boundary.}$$

It is well known that this problem have eigenvalues $\lambda = (k^2 + n^2)\pi^2$ and eigenfunctions $u = \frac{2}{\pi}\frac{1}{\sqrt{k^2+n^2}}\sin k\pi x \sin n\pi y$ (k, n = 1, 2, , 3,……) .

In this case (7) reduces to the linear equations

$$\sum_j u_j\left\{\sum_e \int\left(\frac{\partial\phi_i}{\partial x}\frac{\partial\phi_j}{\partial x}+\frac{\partial\phi_i}{\partial y}\frac{\partial\phi_j}{\partial y}\right)dxdy\right\} = \lambda\sum_j u_j\left\{\sum_e \int \varphi_i\varphi_j\,dxdy\right\}$$

$$i = 1, 2, \cdots\cdots, N.$$

Here, in the case of triangular elements we have the element stiffness matrix as

$$K^e = (k_{ij}^e) = \left[\frac{1}{2A}(b_i b_j + c_i c_j)\right],$$

and in the case of rectangular elements, with uniform mesh $h = x_{i+1} - x_i = y_{j+1} - y_j$, the element stiffness matrix is

$$K^e = (k_{ij}^e) = \frac{1}{6}\begin{pmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{pmatrix}$$

and element mass matrix is

$$M^e = (m_{ij}^e) = \frac{h^2}{36}\begin{pmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{pmatrix}$$

Thus we obtain the matrix eigenvalue problem

$$KU = \lambda MU$$

with

$$K_{ij} = \sum_e k_{ij}^e \qquad \text{and} \qquad M_{ij} = \sum_e m_{ij}^e .$$

Matrices K and M are sparse having only nine nonzero diagonals. We remark that when uniform square nets are used, the finite element scheme is equivalent to the Polya's finite difference scheme.[5)]

Numerical reselts for minimum eigenvalue are as follows.

| Exact $\mu_1 = 1/\lambda_1 = 0.05071$ | | nodes | elements | inner nodes |
|---|---|---|---|---|
| triangelar element (linear polynomial) | | | | |
| h = 1/4 | 0.04443 | 25 | 32 | 9 |
| h = 1/8 | 0.04852 | 41 | 64 | 25 |
| rectangular element | | | | |
| h = 1/4 | 0.08414 | 25 | 16 | 9 |
| h = 1/5 | 0.04903 | 36 | 25 | 16 |
| h = 1/6 | 0.04948 | 49 | 36 | 25 |
| triangular element (quadratic polynomial) | | | | |
| h = 1/4 | 0.05015 | 25 | 8 | 9 |
| h = 1/8 | 0.05034 | 41 | 16 | 25 |

These numerical experiments have been performed on the NEAC 2200-500 at the Tohoku University Computer Center.

**References**

1) A. K. Aziz Ed. The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations. Academic Press (1972) .
2) J. T. Oden : Finite Elements of Nonlinear Continua. McGraw-Hill (1972) .
3) O. C. Zienkiewicz : The Finite Element Method in Engineering Science. McGraw-Hill (1971) .
4) M. H. Schultz : Spline Analysis. Prentice-Hall (1973) .
5) G. E. Forsythe, W. R. Wasow : Finite Difference Methods for Partial Differential Equations. John Wiley (1960) .