Some Generalization of Lyapunov's Stability Theorem

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1. Introduction

It is well-known that Lyapunov has discussed the stability of solutions of a system of differential equations by utilizing a scalar function satisfying some conditions. Lyapunov's second method is a very useful and powerful instrument in discussing the stability of the system of differential equations. Its power and usefulness lie in the fact that the decision is made by investigating the differential equation itself and not by finding solutions of the differential equation. However, it is great difficult to find the Lyapunov function satisfying certain conditions. Therefore, it is important to obtain a weak sufficient condition for a stability theorem.

In this paper, we will state some extension of Lyapunov's stability theorem.

2. Definitions and Notations

We shall first list some notations.

Let I denote the interval $0 \leqq t < \infty$ and R^n denote Euclidian n-space.

For $x \in R^n$, let $\|x\|$ be the Euclidian norm of x, and D is a domain such that $\|x\| \leqq H$, $H > 0$.

We shall denote by $C_0(x)$ the family of functions which satisfy locally a Lipschitz condition with respect to x.

We consider a system of differential equations

$$\frac{dx}{dt} = f(t, x), \tag{1}$$

where x is an n-dimensional vector and $f(t, x)$ is n-dimensional vector function which is defined on a region in $I \times R^n$, and is continuous in (t, x) on $I \times D$.

We assume that $f(t, 0) \equiv 0$.

Throughout this paper a solution through a point (t_0, x_0) in $I \times R^n$ will be denoted by such a form as $x(t; x_0, t_0)$.

We introduce the following definitions.

Definition 1. The equilibrium of the system (1) is said to be stable if for any $\varepsilon > 0$ and any $t_0 \in I$ there exists a $\delta(t_0, \varepsilon) > 0$ such that the inequality $\|x_0\| < \delta$

implies $\| x(t; x_0, t_0) \| < \varepsilon$ for all $t \geqq t_0$.

Definition 2. The equilibrium of the system (1) is said to be uniformly stable if δ of Definition 1 is independent of t_0.

Definition 3. The equilibrium of the system (1) is said to be asymptotically stable if it is stable and if there exists a $\delta_0(t_0) > 0$ such that if $\| x_0 \| < \delta_0(t_0)$, $x(t; x_0, t_0) \to 0$ as $t \to \infty$.

Definition 4. The equilibrium of the system (1) is said to be uniformly asymptotically stable if it is uniformly stable and there exist $\delta_0 > 0$ and $T(\varepsilon) > 0$ for any $\varepsilon > 0$ such that if $\| x_0 \| < \delta_0$, $\| x(t; x_0, t_0) \| < \varepsilon$ for all $t \geqq t_0 + T(\varepsilon)$.

Definition 5. Let $V(t, x)$ be a continuous scalar function defined on an open set, and let $V(t, x) \in C_0(x)$. Corresponding to $V(t, x)$, we define the function

$$V'_{(1)}(t, x) = \overline{\lim_{h \to 0+}} \frac{1}{h} \left\{ V(t+h, x+hf(t, x)) - V(t, x) \right\}.$$

In case $V(t, x)$ has continuous partial derivatives of the first order, it is evident that

$$V'_{(1)}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t, x),$$

where " $\cdot$ " denotes a scalar product.

Definition 6. A function $V(t, x)$ is said to be positive semidefinite if we have $V(t, x) \geqq 0$ in some set $S = \{T \leqq t < \infty, \ \|x\| \leqq B\}$ with $T \geqq t_0$, $0 < B$. A function $W(x)$, independent of t, is said to be positive definite if $W(x) > 0$ for all $x \neq 0$, $\| x \| < B$. A function $V(t, x)$ depending on t, is said to be positive definite if there exists a positive definite function $W(x)$ such that $V(t, x) \geqq W(x)$ in some set S.

Definition 7. A function $V(t, x)$ is said to be bounded if there is a constant $M > 0$ such that $|V(t, x)| < M$ in some set S. A bounded function $V(t, x)$ is said to have an infinitesimal upper bound if, given $\varepsilon > 0$, there exists an $h > 0$ such that $|V(t, x)| < \varepsilon$ for all t and $\| x \| < h$.

Definition 8. A function $V(t, x)$ is called decrescent if the relation $\lim V(t, x) = 0$ for $\| x \| \to 0$ holds uniformly in t.

Definition 9. A real continuous function $\phi(r)$ belongs to class K ($\phi \in K$) if it is defined for $0 \leqq r \leqq r_1 < \infty$, strictly monotone-increasing, and vanishing at $r = 0$.

Definition 10. A real continuous function $\sigma(s)$ belongs to class L ($\sigma \in L$) if it is defined for $0 \leqq s_1 \leqq s < \infty$, strictly decreasing and tending to zero for $s \to \infty$.

If necessary, parameters are written as a second argument, for instance $\phi(\| x \|, t)$.

3. Preliminary Results

Lyapunov's theorems are expressed in various forms caused with the slight differences of the conditions. Now, we classify them the following patterns. For proofs of these theorems, see references.

【Stability】

〖Theorem 1〗 *Suppose that there exists a Lyapunov's function* $V(t, x)$ *defined on* $I \times D$ *which satisfies the following conditions* ;

(i) $V(t, 0) \equiv 0$,

(ii) $a(\|x\|) \leqq V(t, x)$, *where* $a(r)$ *is continuous increasing, positive definite function,*

(iii) $V'_{(1)}(t, x) \leqq 0$.

Then, the equilibrium of the system (1) *is stable.* (cf. [1], [2], [5], [7], [9])

〖Theorem 2〗 *If a function* $V(t, x)$ *exists which is definite and whose derivative* $V'_{(1)}(t, x)$ *is a semidefinite function whose sign contrary to that of* $V(t, x)$, *then the equilibrium of the system* (1) *is stable.* (cf. [3], [4], [6], [11])

〖Theorem 3〗 *The equilibrium of the system* (1) *is stable if there exists a function* $\phi \in K$ *such shat* $\|x(t; x_0, t_0)\| \leqq \phi(\|x_0\|, t_0)$ *for all* $t \geqq t_0$. (cf. [9], [12], [13])

【Uniform Stability】

〖Theorem 4〗 *If condition* (ii) *in Theorem* 1 *is replaced by*

(ii)′ $a(\|x\|) \leqq V(t, x) \leqq b(\|x\|)$, *where* $a(r)$ *and* $b(r)$ *are continuous increasing, positive definite functions, then the equilibrium of the system* (1) *is uniformly stable.* (cf. [1], [2])

〖Theorem 5〗 *The equilibrium of the system* (1) *is uniformly stable with respect to* t_0 *whenever a Lyapunov function* $V(t, x)$ *exists that is positive definite, admits an infinitesimal upper bound, and has a derivative that is negative semidefinite when evaluated on trajectories of* (1). (cf. [5], [7])

〖Theorem 6〗 *The equilibrium of the system* (1) *is uniformly stable whenever a Lyapunov function* $V(t, x)$ *exists that is positive definite and decrescent, and whose derivative along any trajectory sufficiently close to the solution should be nonpositive.* (cf. [8])

〖Theorem 7〗 *The equilibrium of the system* (1) *is uniformly stable if there exists a function* $\Psi \in K$ *such that* $\|x(t; x_0, t_0)\| \leqq \Psi(\|x_0\|)$ *for all* $t \geqq t_0$. (cf. [9], [12], [13])

【Asymptotic Stability】

〖Theorem 8〗 *Under the assumption in Theorem* 1, *if* $V'_{(1)}(t, x) \leqq -c(\|x\|)$, *where* $c(r)$ *is continuous on* $[0, H]$ *and positive definite, and if* $f(t, x)$ *is bounded, then the equilibrium of the system* (1) *is asymptotically stable.* (cf. [1], [10])

〖Theorem 9〗 *The equilibrium is asymptotically stable if there exists a positive definite, decrescent function* V(t, x) *such that its derivative for* (1) *is negative definite.* (cf. [4], [6], [9])

〖Theorem 10〗 *Suppose that there exists a function* V(t, x) *defined for* $t \geqq t_0$, $\|x\| \leqq H$ *continuous, and with the following properties* ;

(i) $V(t, 0) \equiv 0$,

(ii) $V(t, x) \geqq a(\|x\|)$, *where* $a(0) = 0$ *and* a(r) *is continuous and monotone-increasing,*

(iii) $V'_{(1)}(t, x) \leqq -c[V(t, x)]$, *where* c(r) *is continuons and monotone-increasing,* $c(0) = 0$. *Then the equilibrium of the system* (1) *is asymptotically stable.* (cf. [2])

〖Theorem 11〗 *If a function* V(t, x) *exists which is definite, and has an infinitesimal upper bound, if the derivative* $V'_{(1)}(t, x)$ *is also a definite function whose sign is contrary to that of* V(t, x), *then the equilibrium of the system* (1) *is asymptotically stable.* (cf. [3], [11])

〖Theorem 12〗 *The equilibrium of the system* (1) *is asymptotically stable if there are two functions* $\phi \in K$ *and* $\sigma \in L$ *such that*

$$\|x(t; x_0, t_0)\| \leqq \phi(\|x_0\|, t_0)\sigma(t-t_0, t_0) \quad \textit{for all} \quad t \geqq t_0.$$ (cf. [9], [12], [13])

[Uniformly Asymptotic Stability]

〖Theorem 13〗 *Under the same assumptions as in Theorem* 3, *if* $V'_{(1)}(t, x) \leqq -c(\|x\|)$, *where* c(r) *is continuous on* [0, H] *and is positive definite, then the equilibrium of the system* (1) *is uniformly asymptotically stable.* (cf. [1], [2], [5], [8])

〖Theorem 14〗 *Under the same assumptions as in Theorem* 3, *if*

$$V_{(1)}'(t, x) \leqq -cV(t, x),$$

where $c > 0$ *is a constant, then the equilibrium of the system* (1) *is uniformly asymptotically stable.* (cf. [1])

〖Theorem 15〗 *Suppose that there exists a Lyapunov function* V(t, x) *defined on* $t \in I$, $\|x\| \leqq H$, *which satisfies the following conditions* ;

(i) $a(\|x\|) \leqq V(t, x) \leqq b(\|x\|)$, *where* a(r) *and* b(r) *are continuous increasing, positive definite functions.*

(ii) $V'_{(1)}(t, x) + V^*(t, x) \to 0$ *uniformly on* $0 < \eta \leqq \|x\| \leqq H$, *for any* η, *as* $t \to \infty$, *where* $V^*(t, x)$ *is continuous and there exists a continuous function* $c(r) > 0$ *for* $0 < r \leqq H$ *such that* $c(\|x\|) \leqq V^*(t, x)$. *Then, if the equilibrium of the system* (1) *is unique to the right, it is uniformly asymptotically stable.* (cf. [1]).

〖Theorem 16〗 *Suppose that there exists a function* V(t, x) *with the properties*

(i) $a(t, \|x\|) \leqq V(t, x) \leqq b(\|x\|)$,

(ii) $V'_{(1)}(t, x) \leqq -c(t, \|x\|)$,

where b(r) *is continuous, monotone-increasing,* $b(0) = 0$, *and* a(t, r), c(t, r) *are* *continuous and such that for every pair* $0 < \alpha \leqq \beta < H$ *there exists* $\theta(\alpha, \beta) \geqq 0$, $k(\alpha, \beta) > 0$ *such that* $a(t, r) > k(\alpha, \beta)$, $c(t, r) > k(\alpha, \beta)$ for $\alpha \leqq r \leqq \beta$, $t \geqq \theta(\alpha, \beta)$. *Then the equilibrium of the system* (1) *is uniformly asymptotically stable.* (cf. [2])

〖Theorem 17〗 *The equilibrium of the system* (1) *is uniformly asymptotically stable if there are two functions* $\Psi \in K$ *and* $\rho \in L$ *such that the inequality*

$$\|x(t; x_0, t_0)\| \leqq \Psi(\|x_0\|)\rho(t-t_0)$$

holds for all $t \geqq t_0$, *provided the initial point* x_0 *satisfies* $\|x_0\| \leqq H$. (cf. [9], [12], [13])

4. Generalization of Lyapunov's Stability Theorem

〖Theorem〗 *Suppose that there exists a Lyapunov function* $V(t, x)$ *defined on* $I \times D$, *which satisfies the following conditions*;

(i) $V(t, 0) \equiv 0$ *and* $V(t, x)$ *is continuous in* (t, x),

(ii) $a(t, \|x\|) \leqq V(t, x)$, *where the function* $a(t, r)$ *is continuous in* (t, r) *on* $I \times D$, $a(t, 0) \equiv 0$, $a(t, r) > 0$ *for* $r \neq 0$ *and increases monotonically with respect to* t *for each fixed* r,

(iii) $V'_{(1)}(t, x) \leqq 0$.

Then the equilibrium of the system (1) *is stable.*

Proof. Corresponding to any $\varepsilon > 0$, $\varepsilon < H$, we have $a(t, \varepsilon) \leqq V(t, x)$ for $t \in I$ and x such that $\|x\| = \varepsilon$. For a fixed $t_0 \in I$, we can choose a $\delta(t_0, \varepsilon)$ such that $\|x_0\| < \delta$ implies $V(t_0, x_0) < a(t_0, \varepsilon)$, where $\delta(t_0, \varepsilon) < \varepsilon$, because $V(t, 0) \equiv 0$ and $V(t, x)$ is continuous. We suppose that a solution $x(t; x_0, t_0)$ of the system (1) such that $\|x_0\| < \delta$ satisfies $\|x(t_1; x_0, t_0)\| = \varepsilon$, where $t_1 = \inf\{t \mid \|x(t; x_0, t_0)\| = \varepsilon\}$.

From the condition (iii), it follows that $V(t_1, x(t_1; x_0, t_0)) \leqq V(t_0, x_0)$ and hence we have

$$a(t_1, \varepsilon) \leqq V(t_1, x(t_1; x_0, t_0)) \leqq V(t_0, x_0) < a(t_0, \varepsilon).$$

This contradicts the condition (ii), and hence, if $\|x_0\| < \delta(t_0, \varepsilon)$, then $\|x(t; x_0, t_0)\| < \varepsilon$ for all $t \geqq t_0$, that is, the equilibrium of the system (1) is stable.

Example.

In the equation

$$\frac{dx}{dt} = -\frac{x}{t} \qquad (t > 0), \qquad (2)$$

set $V(t, x) = tx^2$ and $a(t, \|x\|) = \frac{1}{2} t\|x\|^2$. $V(t, x)$ satisfies the conditions in our theorem. In fact,

$$V'_{(2)}(t, x) = x^2 + 2tx\frac{dx}{dt} = -x^2 \leqq 0.$$

Therefore, the equilibrium of the system (2) is stable.

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