Some Methods for Eigenvalue Problem

Takashi YOSHIMURA

(Received on 31, Oct., 1972)

1. Introduction

We shall consider the Sturm-Liouville problem

(1.1) $Ly+\lambda ry=(py')'-qy+\lambda ry=0 \qquad a\leq x\leq b,$

(1.2) $\alpha_1 y(a)-\alpha_2 p(a)y'(a)=0, \qquad \beta_1 y(b)+\beta_2 p(b)y'(b)=0,$

with p and r positive, q nonnegative on [a, b], and p piecewise continuously differentiable, p and q piecewise continuous on [a, b]. Also $\alpha_i\geq 0$, $\beta_i\geq 0$.

Since equation (1.1) may be reduced to the form of equation

(1.3) $Ly+\lambda ry=y''-qy+\lambda ry=0$

by a transformation of the variables, we may assume without loss of generality that $p(x)=1$.

It is known that this problem has positive eigenvalues $\lambda_1<\lambda_2$ …… and corresponding eigenfunctions y_1, y_2, ……, which are continuously differentiable and have a continuous second derivative at each point of continuity of r and q.

In the following sections, for the numerical solution, we subdivide the interval $a\leq x\leq b$ by a mesh of points

$$\Delta : a=x_0<x_1< \cdots\cdots <x_N=b,$$

where $x_i=a+ih$ $(i=0, 1, 2, \cdots\cdots, N)$ $h=\dfrac{b-a}{N}$.

2. Spline function method[1]

We seek an interpolating spline function S(x) which is continuous together with its first and second derivatives on [a, b], coinsides with a cubic in each subinterval $x_{j-1}\leq x\leq x_j$ $(j=1, 2, \cdots\cdots, N)$, and satisfies $S(x_j)=y_j$ $(j=0, 1, \cdots\cdots, N)$. Let $S''(x_j)=M_j$ $(j=0, 1, \cdots\cdots, N)$, then

(2.1) $$S(x)=M_{j-1}\frac{(x_j-x)^3}{6h}+M_j\frac{(x-x_{j-1})^3}{6h}+\left(\frac{y_j}{h}-\frac{M_jh}{6}\right)(x-x_{j-1})+\left(\frac{y_{j-1}}{h}-\frac{hM_{j-1}}{6}\right)\cdot(x_j-x)$$

and

(2.2) $$S'(x)=-M_{j-1}\frac{(x_j-x)^2}{2h}+M_j\frac{(x-x_{j-1})^2}{2h}+\frac{y_j-y_{j-1}}{h}-\frac{M_j-M_{j-1}}{6}h.$$

From the continuity of $S'(x)$ at x_j, we have

(2.3) $$\frac{1}{2}M_{j-1}+2M_j+\frac{1}{2}M_{j+1}=3h^{-2}(y_{j-1}-2y_j+y_{j+1}) \quad (j=1, 2, \cdots\cdots, N-1).$$

This is a linear system with N−1 equations in N+1 unknowns M_0, M_1, ……, M_N, therefore two additional conditions are needed. For $S'(a)=y'(a)$ and $S'(b)=y'(b)$ we obtain the relations

(2.4) $$2M_0+M_1=\frac{6}{h}\left(\frac{y_1-y_0}{h}-y'_0\right), \quad M_{N-1}+2M_N=\frac{6}{h}\left(y'_N-\frac{y_N-y_{N-1}}{h}\right).$$

Equations (2.3) and (2.4) are written as

$$
(2.5)\quad \begin{pmatrix} 2 & 1 & 0 & \cdots\cdots & 0 & 0 & 0 \\ 1/2 & 2 & 1/2 & \cdots\cdots & 0 & 0 & 0 \\ 0 & 1/2 & 2 & \cdots\cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots\cdots & 1/2 & 2 & 1/2 \\ 0 & 0 & 0 & \cdots\cdots & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{N-1} \\ M_n \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-1} \\ d_N \end{pmatrix}
$$

where d_j ($j=0, 1, \cdots\cdots, N$) represents the right hand member of (2.3) and (2.4).

For the problem (1.3) with (1.2), let us introduce the cardinal splines, which are a set of N+3 independent splines forminig a basis for all cubic splines on the mesh Δ. We define these in the following way:

$A_k(x)$ ($k=0, 1, \cdots\cdots, N$) and $B_k(x)$ ($k=0, N$) are cubic function on Δ with

(2.6) $A_k(x_j)=\delta_{kj}$ ($j=0, 1, \cdots\cdots, N$), $A'_k(x_i)=0$ ($i=0, N$) $k=0, 1, \cdots\cdots, N$

$B_k(x_j)=0$ ($j=0, 1, \cdots\cdots, N$), $B_k'(x_i)=\delta_{ki}$ ($i=0, N$) $k=0$ and N.

Here δ_{kj} is the Kronecker delta.

We may express the spline satisfying end conditions (2.4) and interpolating on Δ to the solution of the differential equation (1.3) in the form

$$
(2.7)\quad \sum_{j=0}^{N} y(x_j)A_j(x)+y'(a)B_0(x)+y'(b)B_N(x).
$$

Substituting (2.7) into (1.3) we form the equation

$$
(2.8)\quad \sum_{j=0}^{N} y(x_j)LA_j(x)+y'(a)LB_0(x)+y'(b)LB_N(x)+\lambda r\left\{\sum_j y(x_j)A_j(x)+y'(a)B_0(x)+y'(b)B_N(x)\right\}=0.
$$

Thus the ordinates $y(x_j)=y_j$ ($j=0, 1, \cdots\cdots, N$) and the slopes $y'(a)=y_0'$, $y'(b)=y'_N$ satisfy the equations

$$
\begin{pmatrix} -\alpha_2 & \alpha_1 & 0 & \cdots\cdots & 0 & 0 \\ B_0''(x_0) & A_0''(x_0)-q_0 & A_1''(x_0) & \cdots\cdots & A_N''(x_0) & B_N''(x_0) \\ B_0''(x_1) & A_0''(x_1) & A_1''(x_1)-q_1 & \cdots\cdots & A_N''(x_1) & B_N''(x_1) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ B_0''(x_N) & A_0''(x_N) & A_1''(x_N) & \cdots\cdots & A_N''(x_N)-q_N & B_N''(x_N) \\ 0 & 0 & 0 & \cdots\cdots & \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} y'_0 \\ y_0 \\ y_1 \\ \vdots \\ y_N \\ y'_N \end{pmatrix} = -\lambda \begin{pmatrix} 0 & & & & \\ & r(x_0) & & & \\ & & r(x_1) & & \\ & & & \ddots & \\ & & & & r(x_N) \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} y'_0 \\ y_0 \\ y_1 \\ \vdots \\ y_N \\ y_N' \end{pmatrix}
$$

From the Fq. (2.5), we have

$$
\begin{pmatrix} B_0''(x_0) & A_0''(x_0) & A_1''(x_0) & \cdots\cdots & A_N''(x_0) & B_N''(x_0) \\ B_0''(x_1) & A_0''(x_1) & A_1''(x_1) & \cdots\cdots & A_N''(x_1) & B_N''(x_1) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ B_0''(x_{N-1}) & A_0''(x_{N-1}) & A_1''(x_{N-1}) & & A_N''(x_{N-1}) & B_N''(x_{N-1}) \\ B_0''(x_N) & A_0''(x_N) & A_1''(x_N) & \cdots\cdots & A_N''(x_N) & B_N''(x_N) \end{pmatrix} = A^{-1}.
$$

$$
\begin{pmatrix} -6/h & -6/h^2 & 6/h^2 & & \cdots\cdots & 0 & 0 & 0 & 0 \\ 0 & 3/h^2 & -6/h^2 & 3/h^2 & \cdots\cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdots\cdots & 3/h^2 & -6/h^2 & 3/h^2 & 0 \\ 0 & 0 & 0 & & \cdots\cdots & 0 & 6/h^2 & -6/h^2 & 6/h \end{pmatrix}
$$

where A is the coefficient matrix in (2.5).

3. Finite element method [2]

Consider a typical element e (i. e., subinterval) with nodes i, j (for the case of equal intervals, $j=i+1$). The ordinates within an interval have to be uniquely defined by two nodal parameters of the element, which are listed as a vector

(3.1) $\{y\}^e=\begin{Bmatrix} y_i \\ y_j \end{Bmatrix}$

The simplest representation is clearly given by linear polynomial

(3.2) $y=\alpha_1+\alpha_2 x$.

The two constants α_i can be evaluated solving simultaneous equations

(3.3) $\begin{cases} y_i=\alpha_1+\alpha_2 x_i \\ y_j=\alpha_1+\alpha_2 x_j \end{cases}$

and we obtain

(3.4) $y=\frac{1}{x_j-x_i}\Big\{(x_j-x)y_i+(x-x_i)y_j\Big\}$.

It is well known that equation (1.3) with (1.2) is the Euler condition that the integral

(3.5) $J(y)=\int_a^b\Big\{(y')^2+qy^2\Big\}dx-\lambda\int_a^b ry^2dx$

be an extremal.

If the unknown function y is defined, element by element, in the form (3.4) where y_i etc. are the nodal parameters, approximate minimization can be carried out. First, we evaluate the element contribution. For any node we can write, by differentiating Eq. (3.5)

(3.6) $\frac{\partial J^e}{\partial y_i}=2\int\Big\{y'\frac{\partial y'}{\partial y_i}+qy\frac{\partial y}{\partial y_i}-\lambda ry\frac{\partial y}{\partial y_i}\Big\}dx.$

Assuming that q and r are constant in the element e, we have for the whole element

$$\frac{\partial J^e}{\partial\{y\}^e}=\begin{Bmatrix} \frac{\partial J^e}{\partial y_i} \\ \frac{\partial J^e}{\partial y_j} \end{Bmatrix}=\frac{2}{x_j-x_i}\begin{bmatrix} 1+(q-r)\frac{(x_i-x_j)^2}{3} & -1+(q-r)\frac{(x_i-x_j)^2}{6} \\ -1+(q-r)\frac{(x_i-x_j)^2}{6} & 1+(q-r)\frac{(x_i-x_j)^2}{3} \end{bmatrix}\{y\}^e$$

Finaly, assembling of the whole set of minizing equations, we have

$$\frac{\partial J}{\partial\{y\}}=H\{y\}=0$$

with

$$H=\begin{bmatrix} & \cdot & \cdot & \cdot & \cdot & \\ -1+(q-r)_{i-1i}\frac{h^2}{6} & & 2+\Big\{(q-r)_{i-1i}+(q-r)_{ii+1}\Big\}\frac{h^2}{3} & & -1+(q-r)_{ii+1}\frac{h^2}{6} \\ & \cdot & \cdot & \cdot & \cdot & \end{bmatrix}.$$

Here f_{ii+1} represents $f(x^*)$ where x^* is some point in $[x_i,\ x_{i+1}]$.

4. Numerical procedure

In any method which we have described above, the differential problem (1.3) with (1.2) is reduced to the algebraic eigenvalue problem of the form

(4.1) $Ay=\lambda By$.

If the matrix A is nonsingular, equation (4.1) is rewritten in the form

(4.2) $A^{-1}By=\mu y$

where $\mu=\frac{1}{\lambda}$, and we can use the Eberlein method for obtaining the eigenvalue and the corresponding eigenfunction.[3]

For example, we consider the problem

$$(4.3)\quad y''+\frac{\lambda}{(1+x)^2}y=0,\ y(0)=y(1)=0.$$

It is easily seen that the analytical solution is $\lambda_n=\frac{n^2\pi^2}{(\log 2)^2}+\frac{1}{4}$ and

$$y_n=(1+x)^{\frac{1}{2}}\sin\left(n\pi\frac{\log(1+x)}{\log 2}\right)$$

In the spline method the matrices A and B are $(N+3)\times(N+3)$. In the finite element method A and B are $(N-1)\times(N-1)$ tridiagonal matrices of the form

$$A=\begin{pmatrix} & \cdot & \cdot & \cdot & \\ \cdots\cdots & -1 & 2 & -1 & \cdots\cdots \\ & \cdot & \cdot & \cdot & \end{pmatrix}$$

$$B=\begin{pmatrix} & \cdot & \cdot & \cdot & \\ \cdots\cdots & r_{i-1i}\frac{h^2}{6} & (r_{i-1i}+r_{ii+1})\frac{h^2}{3} & r_{ii+1}\frac{h^2}{6} & \cdots\cdots \\ & \cdot & \cdot & \cdot & \end{pmatrix}.$$

Remark In the case of $q(x)=0$ and $y(a)=y(b)=0$, using the spline method Eq. (1.3) can be reduced to

$$M_j+\lambda r_j y_j=0\quad (j=1,\ 2,\ \cdots\cdots,\ N-1).$$

Thus from Eq. (2.3) we obtain

$$\begin{pmatrix} \cdots\cdots & -1 & 2 & -1 & \cdots\cdots \end{pmatrix}\begin{pmatrix} y_{i-1} \\ y_i \\ y_{i+1} \end{pmatrix}=\lambda\begin{pmatrix} \frac{h^2}{6}r_{i-1} & \frac{2h^2}{3}r_i & \frac{h^2}{6}r_{i+1} \end{pmatrix}\begin{pmatrix} y_{i-1} \\ y_i \\ y_{i+1} \end{pmatrix}$$

One should note that this modified equation is very close to that of finite element method.

The numerical result for the smallest eigenvalue and corresponding eigenfunction is as follows:

	spline method (modified method)	finite element method	analytical solution	initial value method
λ	20.98	20.43	20.79	20.799
y_1	0.1630	0.1632	0.1642	0.1643
y_2	0.2998	0.3002	0.3013	0.3013
y_3	0.3947	0.3950	0.3957	0.3958
y_4	0.4417	0.4418	0.4420	0.4420
y_5	0.4422	0.4421	0.4418	0.4417
y_6	0.4018	0.4015	0.4009	0.4008
y_7	0.3286	0.3282	0.3275	0.3274
y_8	0.2315	0.2312	0.2306	0.2305
y_9	0.1193	0.1191	0.1188	0.1187
IT	26 (4)	4	IT: number of iteration for Eberlein	
used time(sec)	38.331 (5.489)	5.297		

These numerical experiments have been performed on the NEAC 2200-500 at the Tohoku

University Computer Center.

5. Initial-value method[4]

Since the procedure which described above have error that is $0(h^2)$, we now consider more higher order accurate approximations. We may reduce the Sturm-Liouville problem (1.3) with (1.2) to initial-value problem:

$$(5.1)\quad \begin{aligned} &Ly+\lambda ry=0, \\ &\alpha_1 y(a)-\alpha_2 y'(a)=0, \quad \gamma_1 y(a)-\gamma_2 y'(a)=1. \end{aligned}$$

If γ_1 and γ_2 are any constants such that $\alpha_2\lambda_1-\alpha_1\gamma_2 \neq 0$, then a unique nontrivial solution of the initial-value problem (5.1) exists. We denote this solution by $y(\lambda;x)$ and consider the equation

$$(5.2)\quad \phi(\lambda)\equiv\beta_1 y(\lambda;b)+\beta_2 y'(\lambda;b)=0.$$

To approximate the solution of the initial-value problem (5.1), we first replace it by an equivalent first-order system such as

$$(5.3)\quad \begin{aligned} &u'(x)=v(x), && u(a)=\frac{\alpha_2}{\alpha_2\gamma_1-\alpha_1\gamma_2} \\ &v'(x)=[q(x)-\lambda r(x)]u(x), && v(a)=\frac{\alpha_1}{\alpha_2\gamma_1-\alpha_1\gamma_2} \end{aligned}.$$

Then the equation (5.2) can be written as

$$(5.4)\quad \phi(\lambda)=\beta_1 u(\lambda;b)+\beta_2 v(\lambda;b)=0.$$

For computing a root of (5.4), we use the Newton's iteration method

$$(5.5)\quad \lambda_n^{(\nu+1)}=\lambda_n^{(\nu)}-\frac{\phi(\lambda_n^{(\nu)})}{\dot\phi(\lambda_n^{(\nu)})} \qquad \nu=0,\ 1,\ 2,\ \cdots\cdots,$$

with $\lambda_n^{(0)}$=arbitrary.

The derivative $\dot\phi(\lambda)$ is given by

$$(5.6)\quad \dot\phi(\lambda)\equiv\beta_1\xi(\lambda;b)+\beta_2\eta(\lambda;b)$$

where ξ and η are the solution of

$$(5.7)\quad \begin{aligned} &\xi'=\eta, && \xi(a)=0 \\ &\eta'=[q(x)-\lambda r(x)]\xi-r(x)u(x), && \eta(a)=0. \end{aligned}$$

Thus while solving the initial-value problem (5.3) numericaly to compute an approximation to $\phi(\lambda_n^{(\nu)})$, we also solve (5.7) numericaly to compute an approximation to $\dot\phi(\lambda_n^{(\nu)})$.

We can perform this numerical procedure with modified Adams method together with the Runge-Kutta method for starting values, which have order of accuracy 4.

For the problem (4.3), the numerical result is shown in above table. we have used the result of finite element method for an initial guess $\lambda_n^{(0)}$.

References

[1] J. H. Ahlberg E. N. Nilson J. L. Walsh : The Theory of Spline and Their Application. Academic Press (1967).

[2] O. C. Zienkiewicz : The Finite Element Method in Engneering Science. McGrawHill. (1971).

[3] P. J. Eberlein : A Jacobi-like Method fof the Automatic Computation of Eigenvalues and Eigen-fuctions of an Arbitrary Matrix. J. Soc. Indust. Appl. Math. 10. 74-88 (1962).

[4] H. B. Keller : Numerical methods for two-point boundary-value problems. Blaisdell (1968).

昭和48年2月