A Note on the Numerical Method of Cascade Flow

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1. Introduction

In singularity method, as seen in ref. [1], [2] and [3], the calculation of the vorticity distribution (i. e. pressure and velocity distribution) of a blade in a cascade have the following procedures.

(I) To derive integral equations as basic relations in flow field, (II) to express the unknown functions in appropriate forms of trigonometric series, (III) to substitute the series into the integral equations and to numerically calculate the integrals which are the coefficients of simultaneous linear equations into which the integral equations are transformed, (IV) to solve this linear equations that results from the control point (pivotal point) processing and (V) to substitute this solution into the series in order to detemine the vorticity distributions.

From these items we can understand that such calculations are fairly troublesome. Circumstanced as these are, a concise method not related to the process (II), (III) and (V) is proposed in this report.

2. Basic Equations

2. 1 Induced Velocity

Consider the cascade whose chord length, solidity, attack angle (concern-ingthe vecfor mean velocity) and stagger angle are denoted by c, c/t, α and β respectively (Fig. 1). On representing the attack angle and blade camber by the vorticity $\gamma(x)$, the induced velocity potential $\varphi(x, y)$ is given by

$$\varphi(x, y) = -\frac{1}{2\pi}\int_0^C \gamma(x') \sum_{n=-\infty}^{\infty} \arctan\left(\frac{y-nt\cdot\cos\beta}{x-nt\cdot\sin\beta-x'}\right) dx' \quad (1)$$

where $\gamma(x)$ is distributed on a chord line instead of a camber line owing to mathematical difficulties. Partial dif-

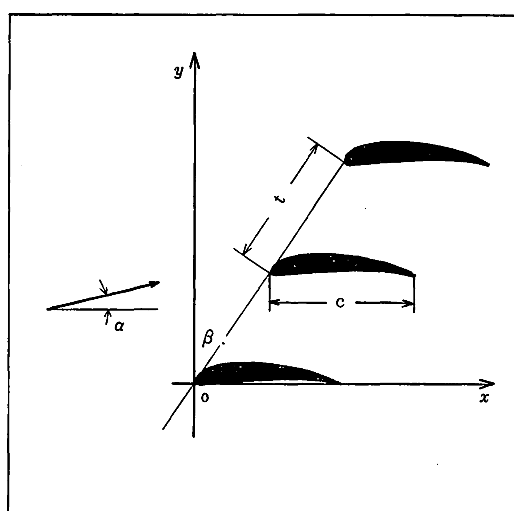

Fig. 1

ferentiation $(\partial/\partial x, \partial/\partial y)$ and limiting process $(y \to o)$ of eq. (1) enable the x and y-component of induced velocity to express as

$$u = -\frac{1}{2\pi}\int_0^C \gamma(x') \sum_{n=-\infty}^{\infty} \frac{nt \cdot \cos\beta}{\{(x-x') - nt \cdot \sin\beta\}^2 + (-nt \cdot \cos\beta)^2} dx' \quad (2)$$

$$v = -\frac{1}{2\pi}\int_0^C \gamma(x') \sum_{n=-\infty}^{\infty} \frac{x - x' - nt \cdot \sin\beta}{\{(x-x') - nt \cdot \sin\beta\}^2 + (-nt \cdot \cos\beta)^2} dx' \quad (3)$$

in which the summation of infinfte series can be readily deformed from mathematical tables and by simple reductions as follows:

$$\sum_{n=-\infty}^{\infty} \frac{x - x' - nt \cdot \sin\beta}{(x-x'-nt \cdot \sin\beta)^2 + (-nt \cdot \cos\beta)^2}$$

$$= \frac{\pi}{t} \frac{\cos\beta \cdot \sinh\left(2\pi\frac{x-x'}{t}\cos\beta\right) + \sin\beta \cdot \sin\left(2\pi\frac{x-x'}{t}\sin\beta\right)}{\cosh\left(2\pi\frac{x-x'}{t}\cos\beta\right) - \cos\left(2\pi\frac{x-x'}{t}\sin\beta\right)} \quad (4)$$

$$\sum_{n=-\infty}^{\infty} \frac{-nt \cdot co\ \beta}{(x-x'-nt \cdot \sin\beta)^2 + (-nt \cdot \cos\beta)^2}$$

$$= \frac{\pi}{t} \frac{\sin\beta \cdot \sinh\left(2\pi\frac{x-x'}{t}\cos\beta\right) - \cos\beta \cdot \sin\left(2\pi\frac{x-x'}{t}\sin\beta\right)}{\cosh\left(2\pi\frac{x-x'}{t}\cos\beta\right) - \cos\left(2\pi\frac{x-x'}{t}\sin\beta\right)} \quad (5)$$

When the right-hand sides of (4) and (5) are written by $\pi R(x, x')/t$ and $\pi I(x, x')/t$, the final forms of induced velocity become

$$u = -\frac{1}{2t}\int_0^C \gamma(x') I(x, x') dx' \qquad v = -\frac{1}{2t}\int_0^C \gamma(x') R(x, x') dx'. \quad (6), (7)$$

2. 2 Integral Equation, Simultaneous Linear Equations

From the tangential flow condition we can get the next equation

$$\frac{dy_f(x)}{dx} = \frac{U \cdot \sin\alpha + v}{U \cdot \cos\alpha + u}, \quad (8)$$

where $y_f(x)$ and U are blade form and vector mean velocity. By substitution of (6) and (7) into (8) we obtain the following singular integral equation;

$$\frac{dy_f}{dx}\left\{\cos\alpha - \frac{1}{2tU}\int_0^C \gamma(x') I(x, x') dx'\right\} = \sin\alpha - \frac{1}{2tU} \times \int_0^C \gamma(x') R(x, x') dx' \quad (9)$$

To solve the above basic equation the integrals on the both sides are expressed in finite series as follows by using the trapezoidal rule.

$$\frac{1}{2tU}\int_0^C \gamma(x') I(x, x) dx' = \sum_{j=1}^{n} \frac{ac}{4ntU} I(i, j) \cdot \gamma(xj) \quad (10)$$

$$\frac{1}{2tU}\int_0^C \gamma(x') R(x, x') dx' = \sum_{j=1}^{n} \frac{ac}{4ntU} R(i, j) \cdot \gamma(xj) \quad (11)$$

Here the notation a has the value 1 for $j=1$ and 2 for $j=2, 3, \cdots, n$. And xj which denotes the end points of the subinterval of integration is defined by c $(j-1)/n$, where n is the number of subintervals (; The interval $0 \leqq x \leqq c$ is subdivided into n equal parts of length c/n). Eq (10) and (11) satisfy the condition (called Kutta-Joukowsky hypothesis) that assures smooth flow at trailing edge. Functions $I(i, j)$ aud $R(i, j)$ are defined by

$$I(i, j) = \frac{\sin\beta\cdot\sinh\{f(i, j)\cdot\cos\beta\} - \cos\beta\cdot\sin\{f(i, j)\cdot\sin\beta\}}{\cosh\{f(i, j)\cdot\cos\beta\} - \cos\{f(i, j)\cdot\sin\beta\}} \tag{12}$$

$$R(i, j) = \frac{\cos\beta\cdot\sinh\{f(i, j)\cdot\cos\beta\} + \sin\beta\cdot\sin\{f(i, j)\cdot\sin\beta\}}{\cosh\{f(i, j)\cdot\cos\beta\} - \cos\{f(i, j)\cdot\sin\beta\}} \tag{13}$$

in which

$$f(i, j) = \frac{\pi c}{nt}\{2(i-j)+1\}. \tag{14}$$

From the relations (9) to (14), we can get a system of linear equations as final form for determining the vorticity distributions and can rewrite in matrix form as follows

$$Aij\,\gamma j = Bi \tag{15}$$

where matrix Aij, row vector γj and column vector Bi are given by

$$Aij = \frac{ac}{4ntU}\{R(i, j) - y_f'(xi)\,I(i, j)\} \tag{16}$$

$$\gamma j = \gamma(xj) \tag{17}$$

$$Bi = \sin\alpha - y_f'(xi)\cdot\cos\alpha \tag{18}$$

respectively, here xi (called control point) is the midpoint of the subinterval and defined as $c(i-0.5)/n$ where i is the positive integer (1, 2, …, n).

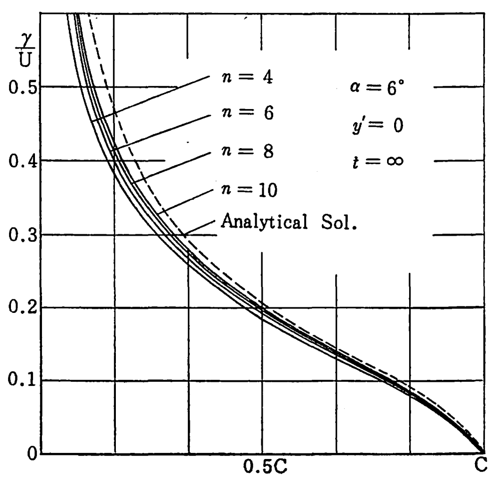

Fig. 2

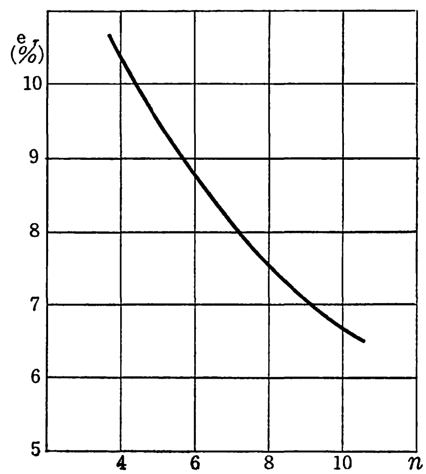

Fig. 3

2. 3 Number of Subintervals, Position of Control Points, Trapezoidal Rule

In our numerical calculus the relation between the number of subintervals of integration n and the accuracy is important. Fig. (2) indicates the effect of n on $\gamma(x)$ in the case of isolated flat plate, comparing with the analytical solution [4] of integral equation. Distribution of er in Fig. 3 is the mean values (from leading edge to trailing edge for 4 to 10 points) of relative error in the same condition as Fig. 2. From these figures it may be concluded the increasing of the number of subintervals assures the better accuracy.

Next we should refer to the position of control points. Three quarter chord theorem proposed by Schlichting [1] is not valid in our treatment from Fig.4 and Fig. 5 in which a dotted line is the result of a 3/4- chord theorem and a dash dotted line a mid-chord approach. Especially in the case of parabolic blade whose camber ratio is 0.05 the quite unexpected results have been obtained (see Fig. 5). Therefore the positions of control points are taken in mid-point of each subinterval and gave the satisfactory results.

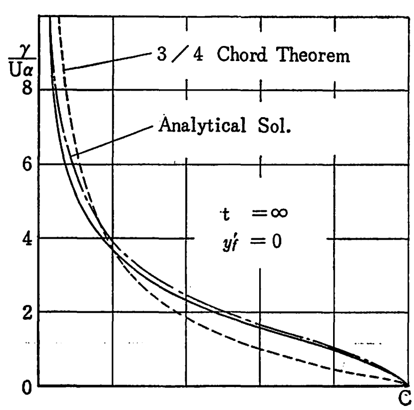

Fig. 4

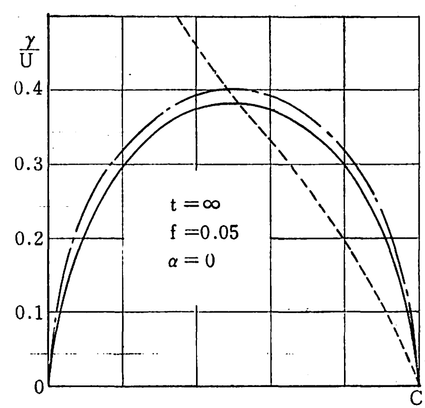

Fig. 5

In the end of this section, the various rules of numerical integration are considered. From Fig. 6 we determined to adopt the trapezoidal rule [5], because for the same number of n this rule gave the best results.

3. Numerical Examples

3. 1 Isolated Blade

In this case the induced velocities are usually small [6], so the equation (9) becomes

$$\frac{dy_f}{dx} = \alpha - \frac{1}{2\pi U}\int_0^C \frac{\gamma(x)}{x-x'}dx'. \quad (19)$$

The expansion for the integral on the

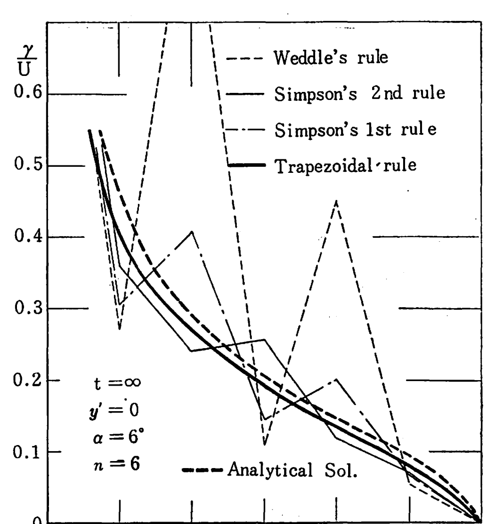

Fig. 6

right-hand side is

$$\int_0^C \frac{\gamma(x')}{x-x'}\,dx' = \sum_{j=1}^{n} \frac{a \cdot \gamma(xj)}{2(i-j)+1}, \qquad (20)$$

so that (19) can be expressed as

$$A'ij\ \gamma'i = B'i \qquad (21)$$

in which $A'ij = a/[2\pi U\{2(i-j)+1\}]$, $\gamma'j = \gamma(xi)$ and $B'i = \alpha - dy_f(xi)/dx$.

Fig. 2 to 5 shown in previous chapter are due to these equations. The values of $\gamma(x)$ of isolated flapped plate are shown in Fig. 7 in which the dotted line is in $\beta' = 0$.

Note ; From this figure to Fig. 11 all the calculations are executed for 10 subintervals.

3. 2 Flat Plate Cascade

The derivative of $y_f(x)$ vanishes, and therefore

$$Aij = \frac{ac}{4ntU} R(i, j) \quad \text{and} \quad Bi = \sin\alpha \qquad (22)$$

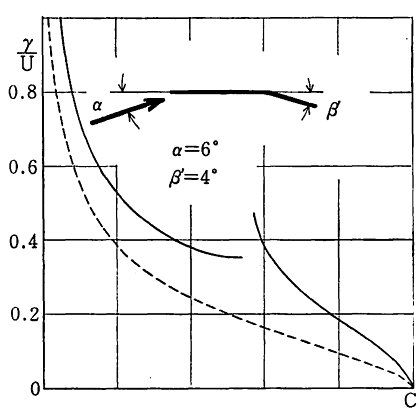

Fig. 7

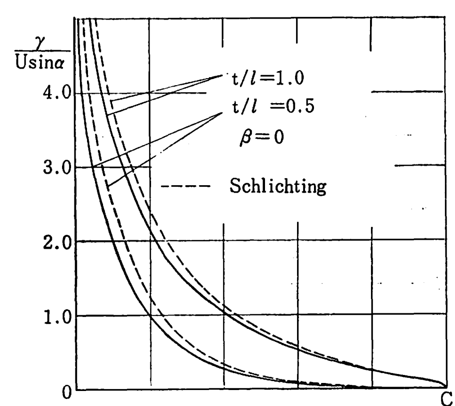

Fig. 8

Fig. 8 shows the comparison of this case with Schlichting's theory. Good agreement can be seen. For various t/c and β, $\gamma(x)/U\alpha$ are given in Fig. 9 and 10.

3. 3 Cascade of Parabolic Blade

Fig. 11 shows the effects of linearization of tangential flow equation (8) on $\gamma(x)$ for parabolic blade cascade. Linearized integral equation

$$y_f'(x) = \alpha - \frac{1}{2tU}\int_0^C \gamma(x')\,R(x, x')\,dx' \qquad (23)$$

is reduced to simultaneous linear equations with

$$Aij = \frac{ac}{4ntU} R(i, j) \text{ and } Bi = \alpha - 4\frac{f}{c^2}\left(1 - 2\frac{xi}{c^2}\right) \qquad (24)$$

in the same way in chap. 2.

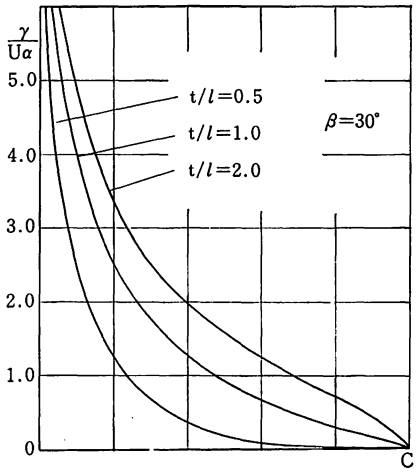

Fig. 9

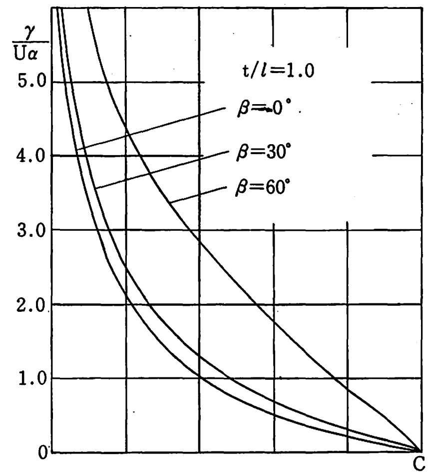

Fig. 10

4. Conclusions

A numerical method based on the trapezoidal rule of integration is developed for determination of vorticity distribution of a blade in a cascade ([7] and [8]). For any form of blade this method is vevy easy to deal with and is specially powerful for the flapped blade.

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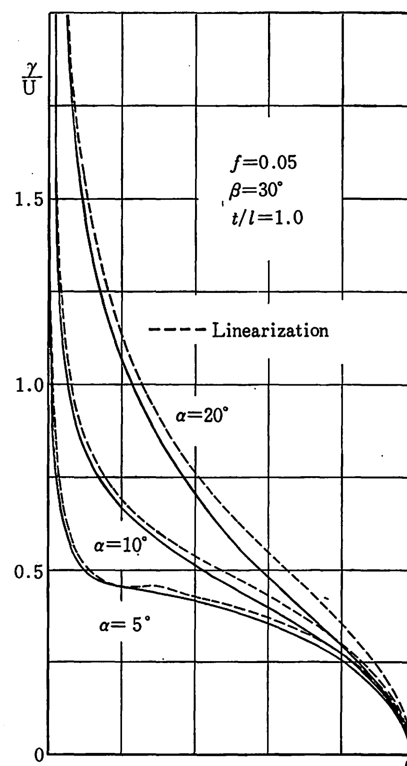

Fig. 11

5. References

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