On the Relations between the Solutions of the Two Differential Equations, $y' = By$ and $w' = 2Bw$

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Let us consider a system of linear differential equations of the first order

$$\frac{dx_i}{dt} = \sum_{k=1}^{n} a_{ik}(t)x_k \qquad (i = 1,2,\cdots\cdots,n) \tag{1}$$

or, if $x = \mathrm{col}(x_1, x_2, \cdots\cdots, x_n)$ and $A(t) = (a_{ij}(t),\ i,j = 1,2,\cdots\cdots,n)$, the matrix equation

$$x' = A(t)x \qquad \left(x' = \frac{d}{dt}x\right) \tag{2}$$

where $a_{ik}(t)$ $(i,k = 1,2,\cdots\cdots,n)$ are real valued functions of a real variable t and piecewise continuous for $0 \leqq t < \infty$.

Let us introduce new functions $y_1, y_2, \cdots\cdots, y_n$ in place of the unknown functions by means of the transformation

$$x_i = \sum_{k=1}^{n} h_{ik}(t)y_k \qquad (i = 1,2,\cdots\cdots,n)\ ,$$

that is

$$x = Hy \tag{3}$$

where $H(t) = (h_{ij}(t),\ i,j = 1,2,\cdots\cdots,n)$ is a Lyapunov matrix.

If Lyapunov transformation (3) carries the system of equations (2) into the system

$$\frac{dy_i}{dt} = \sum_{k=1}^{n} b_{ik}(t)y_k \qquad (i = 1,2,\cdots\cdots,n)\ ,$$

that is

$$y' = By, \tag{4}$$

the null solution of which is stable, asymptotically stable or unstable in the sence of Lyapunov, then the null solution $x_i = 0$ of the initial system (1) possesses the same property.

Therefore the consideration of the stability, asymptotic stability or instability of of (2) is equivalent to that of (4) .

The properties of Lyapunov transformations and Lyapunov matrices may be found in [2] .

Burton [1] has shown that (2) can be mapped into a system (4) such that all solutions of a system

$$w' = 2Bw \tag{5}$$

can be found, and established that, special conditions, the solutions of the two systems, (4) and (5), have stability properties which are essentially the same.

Now we write the matrix A as a sum of piecewise continous matrices

$$A = A_1 + A_2 \tag{6}$$

so that both

$$h' = 2A_1h \tag{7}$$

and

$$z' = 2A_2z \tag{8}$$

can be integrated for principal solution matrices H and Z respectively.

Theorem (Burton) : For H defined by (7), the transformation (3) maps (2) into (4) and

$$z = Hw \tag{9}$$

maps (8) into (5), where $B = H^{-1}[A - H'H^{-1}]H$. Therefore, if H and Z are known, then $W = H^{-1}Z$ is known.

Proof of this theorem is in [1].

Now we are interested in the relations between the solutions of (4) and (5).

Theorem : If there exists a matrix M satisfying that $W = MM^T$ and that $M'M^T$ is symmetric, where T denotes the transpose, then $Y = M$.

Proof. If $M'M^T$ is symmetric, then $(M'M^T)^T = M'M^T$.

Since $(M'M^T)^T = M(M^T)'$, we have

$$M'M^T = M(M^T)' . \tag{10}$$

By (10), we have

$$W' = 2M'M^T = 2M'(M^{-1}M)M^T = 2M'M^{-1}(MM^T) = 2M'M^{-1}W.$$

And we have

$$M'M^{-1} = B \quad \text{or} \quad M' = BM.$$

Thus we have $Y = M$. (Q. E. D)

Burton has shown that $W = YY^T$ if W and B are symmetric from $V' = 0$ setting

$$V = Y^TW^{-1}Y.$$

We have the following result with respect to the assumption of the avove theorem.

Remark : The necessary and sufficient condition in order that $M'M^T$ is symmetric is that $M'M^{-1}$ is symmetric.

Proof. we know that MM^T is symmetric by $(MM^T)^T = (M^T)^TM^T = MM^T$. Therefore, from $M'M^T = (M'M^{-1})(MM^T)$, if $M'M^T$ is symmetric, then $M'M^{-1}$ is also symmetric, and the inverse of this is also true.

Example : Let us consider

$$w' = 2Bw, \text{ where } B = \begin{pmatrix} \cot 2t & -\operatorname{cosec} 2t \\ -\operatorname{cosec} 2t & \cot 2t \end{pmatrix}.$$

If we put

$$M = \begin{pmatrix} \cos t & \sin t \\ \cos t & -\sin t \end{pmatrix},$$

then we have

$$M'M^T = \begin{pmatrix} 0 & -\sin 2t \\ -\sin 2t & 0 \end{pmatrix}$$

and this is symmetric matrix.

And we can easily show that $W = MM^T$ satisfies $W' = 2Bw$.

Therefore $M = \begin{pmatrix} \cos t & \sin t \\ \cos t & -\sin t \end{pmatrix}$ is the solution of the equation $Y' = BY$

Further, $MM^T = \begin{pmatrix} 1 & \cos 2t \\ \cos 2t & 1 \end{pmatrix}$ is symmetric.

and $M'M^{-1} = -\frac{1}{2}\begin{pmatrix} \tan t - \cot t & \tan t + \cot t \\ \tan t + \cot t & \tan t - \cot t \end{pmatrix}$ is also symmetric.

References

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