Application of method solving nonlinear partial differential equation to nonlinear ordinary differential equation

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1. Introduction

The purpose of this report is to discuss the application of a method in the numerical solution of nonlinear parabolic partial differential equation for solving nonlinear ordinary differential equation with split boundary conditions.

We shall be concerned with the boundary value problem

(1.1) $u'' + f(x,u,u') = 0, \quad 0<x<1, \quad u(0) = u(1) = 0,$

and corresponding parabolic partial differential equation

(1.2) $u_{xx} + f(x,u,u_x) = u_t, \quad 0<x<1, \quad 0<t \leqq T,$

$u(0,t) = u(1,t) = 0, \quad 0<t \leqq T, \quad u(x,0)$ specified $0<x<1.$

As a difference analogue of (1.2), we consider the backward difference equation

(1.3) $\Delta^2_x w_{in} + f(x_i, w_{in-1}, \delta_x w_{in-1}) = \nabla_t w_{in}$

where $h = (N+1)^{-1}$, $k = TM^{-1}$; $x_i = ih$, $t_n = nk$, $w_{in} = w(x_i, t_n)$,

$$\Delta^2_x w_{in} = h^{-2}(w_{i+1n} - 2w_{in} + w_{i-1n})$$

$$\delta_x w_{in} = (2h)^{-1}(w_{i+1n} - w_{i-1n})$$

$$\nabla_t w_{in} = k^{-1}(w_{in} - w_{in-1}).$$

In section 3 convergence of the solution of (1.3) to that of (1.2) will be established.

This method of solving partial differential equation is one in which each time step of an unsteady-state problem may be regarded as a stage of the iteration of solving nonlinear ordinary differential equation, with the initial conditions being regarded as the starting values, or, first guess. Thus the time increment k of the unsteady-state problem may be regarded as the iterative parameter of the steady-state problem.

The only consideration of interest in the solution of a nonlinear ordinary differential equation is that convergence to the correct steady-state solution be attained in the least number of iterations. Generaly, the iteration parameter can be increased in size as the solution is approached. Furthermore, since the intermediate values are no interest, the initial iteration parameter can be larger than the initial time step which is used in an unsteady-state problem. This point will be discussed in section 4.

2. Preliminaries

Let R denote the rectangular region defined by the inequalities R: $0<x<1$, $0<t \leqq T$; and R denote the closure of R. The boundary of R is composed of the three segments B_0 $(0 \leqq x \leqq 1,\ t=0)$, B_1 $(x=0,\ 0<t \leqq T)$, B_2 $(x=1,\ 0<t \leqq T)$. Let $\bar{R}_{hk}$ be the rectangular lattice covering $\bar{R}$ determined by the intersections of the coordinate lines $x = ih$,

(i=0, 1,⋯, N+1), t=nk(n=0, 1,⋯,M). The boundary B_{hk} of $\bar{R}_{hk}$ is union of the three sets $B^i_{hk}=B_i\cap\bar{R}_{hk}$ (i=0,1,2). The interior of the lattice is the set $R_{hk}=\bar{R}_{hk}-B_{hk}$.

Furthermore, let w_n represent the column vector $(w_{1n},w_{2n},\cdots w_{Nn})$, and we introduce norms

$$\|w_n\|^2=h\Sigma_{i=1}^{N}|w_{in}|^2 \quad \text{and} \quad \|w_n\|_1^2=h\Sigma_{i=1}^{N+1}|w_{in}|^2.$$

Lemma 1. *Let f and g be nonnegative functions defined on the integers* 1, 2⋯, *M. Let g be nondecreasing. If C is a constant and*

$$f_n\leqq g_n+Ck\,\Sigma_{r=1}^{n-1}f_r \quad (k<0),$$

then

$$f_n\leqq g_n exp(Ckn).$$

Lemma 2. *Let v be a solution of the difference equation*

(2.1) $\nabla_t v=\Delta^2_x v+G \quad$ *in* R_{hk}.

If v vanishes on B_{hk}, *then*

$$\|\nabla_x v_n\|_1^2\leqq 2\,\Sigma_{r=1}^{n}\|G_r\|^2 k$$

for all sufficiently small k.

Lemma 3. *Let v be any function defined on* $\bar{R}_{hk}$ *which vanishes on* B_{hk}. *Then*

$$\max_{\substack{0\leqq i\leqq N+1\\ 0\leqq n\leqq M}}|v_{in}|\leqq\frac{1}{2}\max_{1\leqq n\leqq M}\|\nabla_x v_n\|_1.$$

For the proof of Lemmas 1, 2, 3 see [1].

Remark. Although the expression in Lemma 2 is slightly different from M.Lees, the proof is essentialy the same with his proof.

3. Convergence of solution of difference equation

we shall approximate the solution of the nonlinear parabolic partial differential equation

(3.1) $u_{xx}+f(x, u, u_x)-u_t=0$

by solution of an associated backward difference equation. Let u (x, t) be a four times boundedly differentiable function on $\bar{R}$ which satisfies (3.1) in R. we assume also that the function f(x, u, p) appearing in (3. 1) has at least first bounded derivative with respect to u and p respectively. The solution u (x, t) is approximated by the function w(x, t) defined on $\bar{R}_{hk}$ which agrees with u(x, t) on B_{hk} and satisfies in R_{hk} the backward difference equation

(3.2) $\Delta_x^2 w_{in}+f(x_i, w_{in-1}, \delta_x w_{in-1})-\nabla_t w_{in}=0.$

The difference eqution (3.2) has a very important property that the system leads to linear algebraic equations, and the matrix of the system is of the tridiagonal type and can be handled quite easily by Thomas algorithm. [2]

Theorem. Let the functions f and u satisfy the conditions stated above, then the solution of the system (3.2) *converges uniformly to the solution of* (3.1) *with an error that is* $O(h^2+k)$.

Proof. It follows from our assumptions and the mean value theorem that u satisfies (3.1) with a local error $O(h^2+k)$, that is

(3.3) $\Delta^2{}_x u_{in} + f(x_i, u_{in-1}, \delta_x u_{in-1}) - \Delta_t u_{in} = O(h^2 + k)$.

To prove the theorem, it is necessary to derive a difference equation satisfied by the error $z=u-w$. Subtracting (3.2) from (3.3) leads the difference equation

(3.4) $\Delta^2{}_x z_{in} + f^*_u z_{in-1} + f^*_p \delta_x z_{in-1} - \nabla_t z_{in} = O(h^2 + k)$.

Where the * indicates that the partial derivatives are evaluated at points called for by the mean value theorem. Set

(3.5) $G_{in} = f^*_u z_{in-1} + f^*_p \delta_x z_{in-1} + O(h^2 + k)$.

Then (3.4) becomes

$$\Delta^2{}_x z_{in} = \nabla_t z_{in} + G_{in}$$

which is the form (2.1). The function z_{in} vanishes on B_{hk}. Hence the assumptions of Lemma 2 are satisfied and we may conclude the existence of a constant Q such that

(3.6) $\|\nabla_x z_n\|_1{}^2 \leqq Qk\,\Sigma^n_{r=1}\|G_r\|^2$.

It follows from (3.5) that

(3.7) $\|G_r\|^2 \leqq Q_0(h^2+k)^2 + Q_1\|\nabla_x z_{r-1}\|_1{}^2$,

and using (3.6) and (3.7) leads to the inequality

(3.8) $\|\nabla_x z_n\|_1{}^2 \leqq QQ_0 T(h^2+k)^2 + QQ_1 k\,\Sigma^{n-1}_{r=1}\|\nabla_x z_r\|_1{}^2$.

Applying Lemma 1 to (3.8) yields for all sufficiently small k

(3.9) $\|z_n\|_1{}^2 \leqq QQ_0 T(h^2+k)^2 \exp(QQ_1 T)$.

It follows, finaly, from (3.9) and Lemma 3 that

(3.10) $\max_{\substack{0\leqq i\leqq N+1 \\ 0\leqq n\leqq M}} |u_{in} - w_{in}| = \max_{i,n} |z_{in}| \leqq \frac{1}{2} QQ_0 T (\frac{1}{2} QQ_1 T)(h^2+k)$.

which is the desired result.

4. Iterative parameter

From (3.10) we shall take $k = \rho h^2$, since it can be proved that this is the most efficient choice.[3]

Let u and w be the solutions of (1.1), (1.3) respectively. Let

$$e_{in} = u_i - w_{in}$$

and

$$z_{in} = e_{in} - e_{in+1} = w_{in+1} - w_{in}.$$

Then it follows from (3.2) that z_{in} satisfies the equation

(4.1) $z_{i-1n} - (2+\omega) z_{in} + z_{i+1n} = -\omega z_{in-1} - h^2(f^*_u z_{in-1} + f^*_p \delta_x z_{in-1})$

where

$$\omega = \frac{h^2}{k} = \frac{1}{\rho}.$$

Note that this is the same with (3.4).

we shall write (4.1) in matrix notation

$$Az_n = Bz_{n-1}, \quad \text{or}, \quad z_n = A^{-1}Bz_{n-1},$$

where $A = \left[\cdots 1 \; \underset{\cdots}{\overset{\cdots}{-2-\omega}} \; 1 \cdots\right] = T_N - \omega I$, and $B = \left[\cdots \frac{h}{2} f^*_p \quad -\omega - h^2 f^*_u \quad -\frac{h}{2} f^* \cdots\right]$.

Then, when the spectral radius has small value the convergence to the steady-state solution is rapid.

Since $T_N=\begin{bmatrix} \cdots & \cdots & \\ \cdots 1 & -2 & 1 \cdots \\ & \cdots & \end{bmatrix}$ has characteristic values $-4\sin^2\frac{i\pi}{2(N+1)}$ $(i=1,2,\cdots,N)$ [4], the matrix A has characteristic values $-\omega-4\sin^2\frac{i\pi}{2(N+1)}$. And hence the characteristic values of A^{-1} are $(-\omega-4\sin^2\frac{i\pi}{2(N+1)})^{-1}$. Actual determination of the value of spectral radius of $A^{-1}B$ is difficult, since the matrix B contains f_u and $f_{u'}$ with respect to unknown function u and u'.

5. Numerical examples

Example 1. $u''+0.49(u')^2+1=0$, $u(0)=u(1)=0$.

Since B has the form $-\omega I+B^*$, where $B^*=\begin{bmatrix} & \cdots & \\ \cdots 0.49hu' & 0 & -0.49hu' \cdots \\ & \cdots & \end{bmatrix}$, the spectral radius of $A^{-1}B$ is approximately $\lambda=\omega(\omega+4\sin^2\frac{\pi}{2(N+1)})^{-1}$, thus the convergence is rapid when ω is as small as possible.

Starting with initial guess $u(x)=0.5x(1-x)$, we obtain the approximate values as shown in Table 1. We used $h=0.1$ and time increment k was increased by 10% each time step. Steady-state was reached after iterations as shown in Table 2.

Table 1

x	approximate solution	exact solution
0.1	0.04653	0.04657
0.2	0.08224	0.08230
0.3	0.10748	0.10757
0.4	0.12253	0.12264
0.5	0.12753	0.12764
0.6	0.12253	0.12264
0.7	0.10748	0.10757
0.8	0.08224	0.08230
0.9	0.04653	0.04657

Table 2

ρ	number of iterations	λ
2	21	0.8363
5	14	0.6714
10	10	0.5053
25	7	0.2901
35	6	0.2259
55	5	0.1566
120	4	0.0785
700	3	0.0141
∞	3	0.0000

Example 2. $u''-e^u=0$, $u(0)=u(1)=0$.

Since the matrix $A^{-1}B$ has the form $(T_N-\omega I)^{-1}\begin{bmatrix} & \cdots & \\ \cdots 0 & -\omega+h^2e^u & 0\cdots \\ & \cdots & \end{bmatrix}$, the spectral radius has the smallest value when $\omega-h^2e^u=0$. with $h=0.01$ and initial guess $u(x)=0$, we have obtained the values shown in Table 3. The number of iterations are shown in Table 4.

Table 3

x	approximate solution	exact solution
0.1	−0.041435	−0.041436
0.2	−0.073268	−0.073269
0.3	−0.095799	−0.095800
0.4	−0.109237	−0.109238
0.5	−0.113703	−0.113704
0.6	−0.109237	−0.109238
0.7	−0.095799	−0.095800
0.8	−0.073268	−0.073269
0.9	−0.041435	−0.041436

Table 4

ρ	number of iterations
700	14
3500	6
7000	4
9000	3
10000	2
11000	3
12000	4
35000	5
∞	6

These numerical experiments have been performed on the NEAC 2200−500 at the Tōhoku University Computer Center.

References

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