Application of Momentum Theorem to Supercavitating Cascade Flow (1)

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1. Introduction

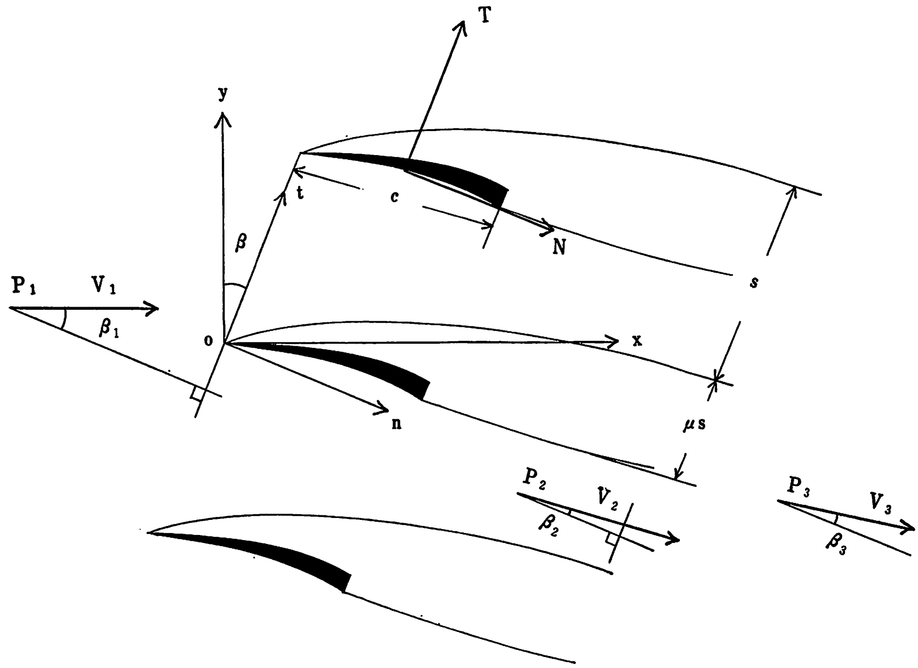
Cavitation occuring in hydraulic machineries makes noise, vibration, large loss in performance and damage and destruction due to erosion. From this point of view the study of supercavitating cascade flow is fundamentally essential for the purpose of speeding up of hydraulic machineries, namely, marine propellers, pumps and hydraulic turbines.

The work in this field is that of Woods [1] who assumed the cusped cavity, Sedov [2] who applied the re-entrant jet model to a flat plate cascade, Nishiyama [3] who developed a cascade flow theory by acceleration potential concepts, Sutherland and Cohen [4] on finite cavity cascade flow, Oba [5] who obtained a clear expression of the drag-lift ratio, Murai and kinoe [6] who assumed the double-spiral model, Jakobsen [7] who used a method of the parametrization by Levi Civita, Cornell [8] on the stall performance of cascades, Acosta [9] on a cascade of circular arc hydrofoils, Grevich [10] who extended Sedov's work and Nishiyama and the author [11] who proposed a singularity method for the supercavitating cascade flow. These two-dimensional potential flow analyses are not so adequate for the real complicated flow in hydraulic machineries, but are the first step towards such a difficult problem.

In this paper the equations for lift, drag and total pressure loss are derived from the momentum theorem of Euler. A theory is developed for two-dimensional, steady, incompressible, finite cavity flow past a straight cascade. Special cases of the equations derived here coincide with those in reference [8].

2. Lift, Drag and Total Pressure Loss

In figure is shown the supercavitating cascade with stagger angle β, pitch s and chord length c. The origin, x-axis and y-axis are assumed to be taken the leading edge of foil, the oncoming entrance flow direction and the normal direction to it. An entrance flow of fluid density ρ, relative velocity V_1 and angle β_1 (which is measured to the normal to the cascade axis direction defined as tangential t-direction) advances with static pressure P_1 and seperates from the foils of cascade at the leading edge, from which a free streamline emanates, forms supercavity and extends to jet flow region, in which the velocity, angle and static pressure are V_2, β_2 and P_2 respectively. Pressure in supercavity is P_c which is usually considered as vapour pressure. The n-component normal to the cascade axis and t-component of the rate of momentum change in a



semi-infinite strip of width t, which extends along the streamline from infinitely far upstream to cavity region, are equal to the n and t-component of the force due to fluid pressure and the blade force acting on the fluid, i.e. -N and -T.

The fluid force acting on blade is follows,

$$N=s\{(P_1-P_2)(1-\mu)+\mu(P_1-P_C)\}+\rho s\{V_1^2\cos^2\beta_1-(1-\mu)V_2^2\cos^2\beta_2\} \tag{1}$$

$$T=\rho s\{V_1^2\cos\beta_1\cdot\sin\beta_1-(1-\mu)V_2^2\cos\beta_2\cdot\sin\beta_2\}, \tag{2}$$

where $s\mu$ is assumed to be the width along cascade axis of far-wake region in dissipative wake model of Joukowsky-Roshko-Eppler. From the definition of cavitation number and equation of Bernoulli, we get

$$P_1-P_2=\frac{1}{2}\rho V_1^2\sigma \tag{3}$$

$$P_1-P_C=\frac{1}{2}\rho V_1^2\left(\frac{1-\lambda^2}{\lambda^2}\right), \tag{4}$$

where the quantity σ is cavitation number and λ is jet velocity ratio V_1/V_2.
Thus

$$N=\frac{1}{2}\rho V_1^2\left(\frac{1-\lambda^2}{\lambda^2}\right)s(1-\mu)+\frac{1}{2}s\mu\rho V_1^2\sigma+\rho sV_1^2\cos^2\beta_1-\rho s(1-\mu)V_2^2\cos^2\beta_2. \tag{5}$$

The last term of this equation, by mass balance

$$\rho sV_1\cos\beta_1=\rho s(1-\mu)V_2\cos\beta_2, \tag{6}$$

can be expressed as

$$\rho s\frac{1}{1-\mu}V_1^2\cos^2\beta_1.$$

Therefore (5) is

$$N=\frac{1}{2}\rho V_1^2c\left(\frac{s}{c}\right)\left\{\left(\frac{1-\lambda^2}{\lambda^2}\right)(1-\mu)+\mu\sigma+2\cos^2\beta_1\left(\frac{\mu}{\mu-1}\right)\right\}. \tag{7}$$

Similarly (2) becomes

$$T=\frac{1}{2}\rho V_1^2c\left(\frac{s}{c}\right)\left\{2\cos\beta_1\left(\sin\beta_1-\frac{\sin\beta_2}{\lambda}\right)\right\}. \tag{8}$$

When the coefficients of N and T are defined as

$$C_N=\frac{N}{\frac{1}{2}\rho V_1^2c}$$

$$C_T=\frac{T}{\frac{1}{2}\rho V_1^2c},$$

from (7) and (8)

$$C_N=\left(\frac{s}{c}\right)\left\{\left(\frac{1-\lambda^2}{\lambda^2}\right)(1-\mu)+\mu\sigma+2\cos^2\beta_1\left(\frac{\mu}{\mu-1}\right)\right\} \tag{9}$$

$$C_T=2\left(\frac{s}{c}\right)\cos\beta_1\left(\sin\beta_1-\frac{\sin\beta_2}{\lambda}\right). \tag{10}$$

When the coefficients of lift L and drag D are given by

$$C_L=\frac{L}{\frac{1}{2}\rho V_1^2c}$$

$$C_D=\frac{D}{\frac{1}{2}\rho V_1^2c},$$

the combination of C_L and C_D with C_N and G_T produces

$$C_L=C_T\cos\beta-C_N\sin\beta$$

$$C_D=C_T\sin\beta+C_N\cos\beta.$$

Consequently

$$C_L=\frac{s}{c}\left[2\cos\beta\cdot\cos\beta_1\left(\sin\beta_1-\frac{\sin\beta_2}{\lambda}\right)-\sin\beta\left\{\left(\frac{1-\lambda^2}{\lambda^2}\right)(1-\mu)+\mu\sigma+2\cos^2\beta_1\left(\frac{\mu}{\mu-1}\right)\right\}\right] \tag{11}$$

$$C_D=\frac{s}{c}\left[2\sin\beta\cdot\cos\beta_1\left(\sin\beta_1-\frac{\sin\beta_2}{\lambda}\right)+\cos\beta\left\{\left(\frac{1-\lambda^2}{\lambda^2}\right)(1-\mu)+\mu\sigma+2\cos^2\beta_1\left(\frac{\mu}{\mu-1}\right)\right\}\right]. \tag{12}$$

In order to simplify the above equations and be able to compare them with the reference [8] the equations (11) and (12) are, by simple calculation, led to

$$C_L=\frac{s}{c}\left[2\cos\beta_1\left\{\sin(\beta_1-\beta)-\frac{1}{\lambda}\sin(\beta_2-\beta)\right\}-\sin\beta\left\{\left(\frac{1-\lambda^2}{\lambda^2}\right)(1-\mu)+\mu\sigma\right\}\right] \tag{13}$$

$$C_D=\frac{s}{c}\left[2\cos\beta_1\left\{\cos(\beta_1-\beta)-\frac{1}{\lambda}\cos(\beta_2-\beta)\right\}+\cos\beta\left\{\left(\frac{1-\lambda^2}{\lambda^2}\right)(1-\mu)+\mu\sigma\right\}\right]. \tag{14}$$

Let the control surface take between jet stream and down stream in a strip width t, then the n-component of momentum equation is given by

$$\rho sV_3^2\cos^2\beta_3-\rho s(1-\mu)V_2^2\cos^2\beta_2=s(1-\mu)P_2+\mu sP_C-sP_3, \tag{15}$$

where suffix 3 means infinitely down stream quantities. From (3) and (4) the right of (15) is

$$s(P_2-P_3)+\mu s\left\{\frac{1}{2}\rho V_1^2\left(\frac{1-\lambda^2}{\lambda^2}\right)-\frac{1}{2}\rho V_1^2\sigma\right\}.$$

Therefore the static pressure drop can be written as

$$P_2-P_3=\mu\left\{\frac{1}{2}\rho V_1^2\sigma-\frac{1}{2}\rho V_1^2\left(\frac{1-\lambda^2}{\lambda^2}\right)\right\}$$
$$+\rho V_3^2\cos^2\beta_3-\rho(1-\mu)V_2^2\cos^2\beta_2. \quad (16)$$

The relation

$$\cos\beta_2=\frac{1}{1-\mu}\frac{V_3}{V_2}\cos\beta_3 \quad (17)$$

which is derived from continuity of mass leads (16) to

$$P_3-P_2=\rho V_3^2\cos^2\beta_3\left(\frac{\mu}{1-\mu}\right)-\frac{1}{2}\rho V_1^2\mu\left(\sigma-\frac{1-\lambda^2}{\lambda^2}\right), \quad (18)$$

while from the fact that the t-component of velocity is unchanged in the mixing process and the mass balance (17) we obtain, after little reduction, total pressure diffence

$$P_{T2}-P_{T3}=\frac{1}{2}\rho V_3^2\Big[\sin^2\beta_3\left\{1+\frac{\cot^2\beta_3}{(1-\mu)^2}\right\}-1$$
$$+2\cos^2\beta_3\left(\frac{\mu}{\mu-1}\right)\Big]+\frac{1}{2}\rho V_1^2\mu\left(\sigma-\frac{1-\lambda^2}{\lambda}\right), \quad (19)$$

$$\text{where } P_{T2}-P_{T3}=\frac{1}{2}\rho V_2^2-\frac{1}{2}\rho V_3^2+P_2-P_3. \quad (20)$$

The equation in square bracket of (19) becomes

$$\cos^2\beta_3\left\{\frac{\mu^2}{(1-\mu)^2}\right\},$$

and consequently the total pressure loss is

$$P_{T1}-P_{T3}=\frac{1}{2}\rho V_3^2\cos^2\beta_3\frac{\mu^2}{(1-\mu)^2}+\frac{1}{2}\rho V_1^2\mu\left(\sigma-\frac{1-\lambda^2}{\lambda^2}\right), \quad (21)$$

since $P_{T1}=P_{T2}$,

where

$$P_{T1}=\frac{1}{2}\rho V_1^2+P_1.$$

From (21) the total pressure loss coefficient defined as

$$\nu=\frac{P_{T1}-P_{T3}}{\frac{1}{2}\rho V_1^2}$$

is

$$\nu=\frac{\mu^2}{(1-\mu)^2}\cos^2\beta_1+\mu\left(\sigma-\frac{1-\lambda^2}{\lambda^2}\right), \quad (21)'$$

where $V_1^2\cos^2\beta_1=V_3^2\cos^2\beta_3$

is used.

Note ; (13) and (14) become simple a little as $\beta=\beta_1$.

3. Conclusions

The equations for lift, drag and total pressure loss are derived from Euler's momentum theorem under the assumptions which are two dimensional, steady, incompressible in the whole flow field and further inviscid between oncoming and jet flow region.

In reference [8] it is concluded the theory is not adequate for low solidity cascade with thick and large cambered foils. But in this paper there are no such restrictions.

Let (13) compares with the equations derived in reference [8] whose expression is

$$C_L=\left\{\frac{\cos\beta_\infty}{\cos\beta_1}\right\}^2\left(\frac{s}{c}\right)\left[\sin\beta\left(\frac{1-\lambda^2}{\lambda^2}\right)+2\cos\beta_1\left\{\sin(\beta_1-\beta)-\frac{1}{\lambda}\sin(\beta_2-\beta)\right\}\right], \quad (22)$$

where the above coefficient is refered to vector mean velocity and direction V_∞, β_∞. The difference between (13) and (22) is the additional term in (13). That is

$$\frac{s}{c}\sin\beta\cdot\mu\left\{\left(\frac{1-\lambda^2}{\lambda^2}\right)+\sigma\right\}.$$

Total pressure loss in reference [8] is given in following expression

$$P_{t1}-T_{t3}=\frac{1}{2}\rho V_3^2\cos^2\beta_3\left\{\frac{\mu}{(1-\mu)^2}\right\}^2. \quad (23)$$

The last term of (21) does not appear in (23).

These are caused by the assumption that in reference [8] the flow seperates to form large eddying wakes and within the wakes the static pressure is taken as the jet stream pressure. Therefore if one could accept the cavity pressure as the jet stream pressure, (13) and (21) would be reduced to (22) and (23). However in supercavitating flow these pressure values do not coincide with each other. The disagreement in these equations are considerable. The qeuations (13), (14) and (21) will be numerically estimated in the second report.

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