A closed 2-form on the quotient space $\mathrm{Hom}(\pi_1(\Sigma_g), SO(4))/SO(4)$

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Following the idea of A. Weinstein [5], we construct a closed 2-form on $\mathrm{Hom}(\pi_1(\Sigma_g), SO(4))$ which is the pullback of a closed 2-form on the quotient space $\mathrm{Hom}(\pi_1(\Sigma_g), SO(4))/SO(4)$.

1 Introduction

In classical theory, it is well-known that there is one-to-one correspondence between the conjugate classes of homomorphisms $\pi_1(M) \to SO(4)$ and the isomorphism classes of flat $SO(4)$-bundles over M. So it is important to investigate the property of $\mathrm{Hom}(\pi_1(M), SO(4))/SO(4)$ in the study of flat $SO(4)$-bundles.

On the other hand, for any Lie group G, we can construct a simplicial manifold NG called nerve of G and the de Rham complex $\Omega^*(NG(*))$ on it. We call this complex the BSS complex. In [5], A. Weinstein introduced the equivariant BSS complex $\Omega^*_{SU(2)}(NSU(2))$ and used a cocycle in it to construct a symplectic form on $\mathrm{Hom}(\pi_1(\Sigma_g), SU(2))/SU(2)$.

In this paper, we construct a closed 2-form on $\mathrm{Hom}(\pi_1(\Sigma_g), SO(4))$ using a cocycle in $\Omega^*_{SO(4)}(NSO(4))$.

2 The Euler class in the BSS complex

In this section we take $G = SO(4)$ and recall a cocycle in $\Omega^4(NG)$ which represents the Euler class.

Theorem 2.1 ([3]). *The cocycle which represents the Euler class of $ESO(4) \to BSO(4)$ in $\Omega^4(NSO(4))$ is a sum of the following $E_{1,3}$ and $E_{2,2}$:*

$$\begin{array}{ccc} E_{1,3} \in \Omega^3(G) & \xrightarrow{d'} & \Omega^3(NG(2)) \\ & & \uparrow{\scriptstyle d} \\ & & E_{2,2} \in \Omega^2(NG(2)) \end{array}$$

$$E_{1,3} = \frac{1}{192\pi^2} \sum_{\tau \in \mathfrak{S}_4} \mathrm{sgn}(\tau)\big((h^{-1}dh)_{\tau(1)\tau(2)}(h^{-1}dh)^2_{\tau(3)\tau(4)} + (h^{-1}dh)_{\tau(3)\tau(4)}(h^{-1}dh)^2_{\tau(1)\tau(2)}\big),$$

$$E_{2,2} = \frac{-1}{64\pi^2} \sum_{\tau \in \mathfrak{S}_4} \mathrm{sgn}(\tau)\big((h_1^{-1}dh_1)_{\tau(1)\tau(2)}(dh_2h_2^{-1})_{\tau(3)\tau(4)} + (h_1^{-1}dh_1)_{\tau(3)\tau(4)}(dh_2h_2^{-1})_{\tau(1)\tau(2)}\big).$$

3 A cocycle in the equivariant BSS complex

In this section we recall a cocycle in $\Omega^4_{NSO(4)}(NSO(4))$.

We take a cochain $\mu \in (\Omega^1(G) \otimes \mathcal{G}^*)^G$ as follows:

$$\begin{aligned}
&\mu(X) = \\
&\frac{-1}{64\pi^2} \sum_{\tau \in \mathfrak{S}_4} \mathrm{sgn}(\tau)\big((X)_{\tau(1)\tau(2)}(h^{-1}dh)_{\tau(3)\tau(4)} \\
&\qquad +(X)_{\tau(3)\tau(4)}(h^{-1}dh)_{\tau(1)\tau(2)}\big) \\
&-\frac{1}{64\pi^2} \sum_{\tau \in \mathfrak{S}_4} \mathrm{sgn}(\tau)\big((X)_{\tau(1)\tau(2)}(dhh^{-1})_{\tau(3)\tau(4)} \\
&\qquad +(X)_{\tau(3)\tau(4)}(dhh^{-1})_{\tau(1)\tau(2)}\big).
\end{aligned}$$

Here $X \in \mathcal{G} = \mathfrak{so}(4)$.

Theorem 3.1 ([4]). *$E_{1,3}+E_{2,2}+\mu$ is a cocycle in $\Omega^4_{SO(4)}(NSO(4))$.*

4 A closed 2-form on $\mathrm{Hom}(\pi_1(\Sigma_g), SO(4))$

In this section, we construct a closed 2-form on $\mathrm{Hom}(\pi_1(\Sigma_g), SO(4))$.

We set $\pi := \pi_1(\Sigma_g)$. The evaluation mapping

$$\mathrm{ev} : \pi^p \times \mathrm{Hom}(\pi, SO(4)) \to SO(4)^p$$

induces a pullback $\mathrm{ev}^* : \Omega^*_{SO(4)}(SO(4)^p) \to \Omega^*_{SO(4)}(\pi^p \times \mathrm{Hom}(\pi, SO(4)))$. Since π is discrete, $\Omega^*_{SO(4)}(\pi^p \times \mathrm{Hom}(\pi, SO(4)))$ is identified with $C^p(\pi) \otimes \Omega^*_{SO(4)}(\mathrm{Hom}(\pi, SO(4)))$, where $C^p(\pi)$ is the space of the real-valued functions on π^p. Especially, $\mathrm{ev}^*E_{2,2}$ belongs to $C^2(\pi) \otimes \Omega^2_{SO(4)}(\mathrm{Hom}(\pi, SO(4)))$.

Proposition 4.1. *We take a 2-cycle $c \in C_2(\pi)$, then $\mathrm{ev}^*E_{2,2}(c)$ belongs to $\Omega^2_{SO(4)}(\mathrm{Hom}(\pi, SO(4)))$ and the following equations hold:*

$$d(\mathrm{ev}^*E_{2,2}(c)) = 0, \qquad d_G(\mathrm{ev}^*E_{2,2}(c)) = 0.$$

So $\mathrm{ev}^*E_{2,2}(c)$ is a closed 2-form and also the pullback of a closed 2-form on the quotient space $\mathrm{Hom}(\pi_1(\Sigma_g), SO(4))/SO(4)$.

Remark 4.1. When c is a 2-boundary, the equation $\mathrm{ev}^*E_{2,2}(c) = 0$ holds so paring $c \in H_2(\pi)$ with $\mathrm{ev}^*E_{2,2}$ defines a natural homomorphism $H_2(\pi) \to \Omega^2_{SO(4)}(\mathrm{Hom}(\pi, SO(4)))$.

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